Problem set 2 (due Thursday February 19th)

1. Starting from the full expression for the radial temperature profile of an irradiated disk,

\[
\left( \frac{T_{\text{disk}}}{T_*} \right)^4 = \frac{1}{\pi} \left[ \sin^{-1} \left( \frac{R_*}{r} \right) - \left( \frac{R_*}{r} \right) \sqrt{1 - \left( \frac{R_*}{r} \right)^2} \right],
\]

Taylor expand the terms on the right hand side to show that for \( r/R_* \gg 1 \), \( T_{\text{disk}} \propto r^{-3/4} \), as we asserted in class.

2. Consider a disk with mass \( M_{\text{disk}} \sim \pi r^2 \Sigma \) and thickness \( h \), at radius \( r \) from a star of mass \( M_* \). By approximating the self-gravity of the disk as that of an infinite sheet, estimate the minimum \( \Sigma \) such that disk self-gravity dominates the vertical acceleration at \( z = h \). Hence, show that,

\[
\frac{M_{\text{disk}}}{M_*} > \left( \frac{h}{r} \right),
\]

is a rough condition for when self-gravity matters for the vertical structure.

3. A disk is vertically isothermal, with a profile,

\[
\rho(z) = \rho_0 \exp[-z^2/2h^2],
\]

as usual. The mid plane Keplerian velocity is \( \Omega_K = \sqrt{GM_*/r^3} \). Suppose that the disk has a radial variation of surface density and temperature,

\[
\Sigma \propto r^{-\gamma} \\
T \propto r^{-\beta}
\]

with \( \gamma \) and \( \beta \) constants. In cylindrical co-ordinates, the condition for hydrostatic equilibrium is,

\[
r\Omega_g^2 = \frac{GM_*}{(r^2 + z^2)^{3/2}} + \frac{1}{\rho} \frac{\partial P}{\partial r},
\]

where \( \Omega_g \) is the gas angular velocity. Find the lowest order expression for the gas angular velocity in the form \( \Omega_g = \Omega_K [1 - ...] \), where the departure from Keplerian rotation is a function of \( (h/r), (z/h) \) and the constants \( \beta \) and \( \gamma \).