Problem Set #2: Solutions

(1) Formula for angular size of Einstein ring is:

$$\theta_e = \frac{2}{cN} \sqrt{\frac{GM \, ds}{dl \, ds}}$$

... where the distances are defined via:

source \hspace{1cm} \text{lens} \hspace{1cm} \text{observer}

\[ dl = x \, ds \]
\[ dls = (1-x) \, ds \]

\[ \Rightarrow \quad A = \pi \, d_l^2 \, \theta_e^2 = \frac{4\pi \, GM \, d_l \, ds}{c^2} \]
\[ = \text{const} \times x \, (1-x) \]

\[ A = \text{const} \times (x - x^2) \]
\[ \frac{dA}{dx} = \text{const} \times (1 - 2x) = 0 \]

\[ \Rightarrow \quad x_{\text{max}} = \frac{1}{2} \]
(2) Mass function for $M > M_0$:

$$\mathcal{f}(M) = 5_0 M^{-2.35}.$$ 

Let mass of a co-rotating white dwarf be $M_{WD}$.

Mass in remnant white dwarfs:

$$M = M_{WD} \times \text{number of white dwarfs}$$

$$= M_{WD} \int_{M_0}^{8M_0} \mathcal{f}(M) \, dM$$

$$= M_{WD} 5_0 \left[ \frac{M^{1.35}}{1.35} \right]_{M_0}^{8M_0}$$

Total mass originally in stars:

$$M_{\text{total}} = \int M \mathcal{f}(M) \, dM = 5_0 \left[ \frac{M^{-0.35}}{-0.35} \right]_{M_0}^{\infty}$$

Fractions locked up in WDs:

$$f = \frac{M}{M_{\text{total}}} = \frac{M_{WD} \times 0.35}{1.35} \left( \frac{M^{1.35}}{M_0^{1.35}} \right)_{M_0}^{8M_0}$$

For $M_{WD} = 0.6 M_0$

$$f = 0.6 \times 0.35 \left[ \frac{8^{-1.35}}{1.35} - 1 \right]$$

$$f = 0.146$$
(3) Mass density near the Galactic center: \( \rho = \rho_0 r^{-\frac{7}{4}} \)

(a) Total mass within 2pc is \( 4.5 \times 10^6 M_0 \)

\[
2\text{pc} = 6.172 \times 10^{18} \text{cm} \\
4.5 \times 10^6 M_0 = 8.9505 \times 10^{39} \text{g.}
\]

\[
M = \int 4\pi r^2 \rho \, dr
\]

\[
= \int 4\pi \rho_0 r^{\frac{7}{4}} \, dr
\]

\[
= 4\pi \rho_0 \left[ \frac{r^{\frac{11}{4}}}{\frac{11}{4}} \right]_0^{2\text{pc}}
\]

\[
\Rightarrow \rho_0 = 2.89 \times 10^{15} \text{ g cm}^{-\frac{7}{4}}
\]

(b) This is just \( v = \sqrt{\frac{GM_\odot}{r}} \)

(2)

(c) Formula for \( \frac{dV}{dt} \) from the lectures is:

\[
\frac{dV}{dt} = -\frac{4\pi G^2 M}{V^2} \text{ nm ln L}
\]

where \( M \) is the massive object, moving with velocity \( v \) with respect to less massive stars of mass \( m \) and number density \( n \).

\[
\ln \Lambda = n \left( \frac{b_{\max}}{b_{\min}} \right) \text{ limit on impact parameter} b.
\]
\[ E_{\text{friction}} = \frac{V}{|dV/dt|} \propto \frac{V^3}{nM} \]

\[
nM = \rho \alpha \frac{\pi r^2}{2} \propto \frac{r^{-3/4}}{r^{-1/2}} \Rightarrow E_{\text{friction}} \propto \frac{r^{-3/4}}{r^{-1/2}} \propto r^{-1/4}.
\]

Now substitute numbers:

\[ E_{\text{friction}} = \frac{V^3}{4\pi G^2 M \rho \ln \Lambda} \]

\[ M = 10 M_\odot = 1.989 \times 10^{33} \text{ g} \]

\[ \rho = \rho_0 \left( \frac{1 \text{ pc}}{1} \right)^{-2/4} = 1.272 \times 10^{-13} \text{ g cm}^{-3} \]

\[ V = 1.39 \times 10^7 \text{ cm s}^{-1} \]

\[ \ln \Lambda = 10. \]

\[ \Rightarrow E_{\text{friction}} = 6.0 \times 10^5 \text{ yr}. \]
Problem Set #3: Solutions

\[ \oint g \cdot d\mathbf{A} = -4\pi G M_{\text{in}} \]

For a spherically symmetric mass distribution \( g = (-g_r, 0, 0) \) in polar coordinates, we have:

\[ g_r \cdot 4\pi r^2 = 4\pi G M_{\text{in}} \]

\[ g_r = \frac{G M_{\text{in}}}{r^2} \]

... as expected.

Consider a cylinder extending from \( -2 \) to \( 2 \) with cap area \( d\mathbf{A} \). By symmetry, \( g = 0 \) except at end caps, where it equals \( g_2 \).

Applying formula above \( g_2 \) (taking positive downwards):

\[ 2g_2 d\mathbf{A} = 4\pi G \cdot 2 z d\mathbf{A} \rho \]

\[ g_2 = 4\pi G \rho z \]
Since $g_2$ is directed downwards, equation of motion is:

$$\frac{d^2 z}{dt^2} = -4\pi G \rho z$$

$$\frac{d^2 z}{dt^2} + k^2 z = 0 \quad \text{with} \quad k = 4\pi G \rho.$$  

The general solution will be sum of sin and cos terms.

E.g., trial solution:

\[
\begin{align*}
z &= \sin (\omega t) \\
\dot{z} &= \omega \cos (\omega t) \\
\ddot{z} &= -\omega^2 \sin (\omega t)
\end{align*}
\]

\[\implies -\omega^2 \sin (\omega t) + k \sin (\omega t) = 0\]

\[\omega = k^{\frac{1}{2}}.\]

**Solution**

\[z = A \sin \left( \left( 4\pi G \rho \right)^{\frac{1}{2}} t \right) + B \cos \left( \left( 4\pi G \rho \right)^{\frac{1}{2}} t \right)\]

For one oscillation: \(\omega t = 2\pi\)

\[\implies \quad P = \frac{2\pi}{\left( 4\pi G \rho \right)^{\frac{1}{2}}} = \left( \frac{\pi}{G \rho} \right)^{\frac{1}{2}}\]

If \(\rho = 0.1 \, M_\odot \, p^{-3} = 6.77 \times 10^{-24} \, \text{g/cm}^3\)

\[\implies \quad P = 2.6 \times 10^{15} \, \text{s} = 8.35 \times 10^9 \, \text{yr} \]
Problem Set 14 - Solutions

1. The Eddington limit is:

\[ L_{\text{Edd}} = \frac{4\pi G M_p}{\sigma_T} M \]

...where \( \sigma_T \) is the Thomson cross-section for electron-photon scattering.

If \( L = \gamma \frac{hc^2}{M} \) with \( \gamma = \text{const.} \),

\[ \dot{M} = \frac{1}{\gamma c^2} \frac{4\pi G M_p}{\sigma_T} M \]

i.e.

\[ \frac{dM}{dt} = kM \quad \text{with} \quad k = \frac{4\pi G M_p}{\gamma c^2 \sigma_T} \]

Solving this equation:

\[ \int \frac{dM'}{M'} = \int k dt' \]

\[ \ln \left[ \frac{M}{M_0} \right] = k t \]

\[ M = M_0 e^{kt} = M_0 e^{t/T} \]

where \( T = k^{-1} = \frac{\gamma c^2 \sigma_T}{4\pi G M_p} \)

Numerically, if \( \gamma = 0.1 \),

\[ T = 1.4 \times 10^5 \text{ s} = 4.5 \times 10^7 \text{ yr} \]
Note: Strictly need to account for fact that radiated energy does not add to rest mass of black hole. In expression for $t$ this means $\eta \rightarrow \eta/(1 - \eta)$. But this correction is not too important.

Time to grow from $10 M_\odot \rightarrow 10^9 M_\odot$ by Eddington limited accretion is then:

$$\ln \left[ \frac{10^9}{10} \right] \times 2 = 18.4 \Rightarrow t$$

$$= 8.3 \times 10^8 \text{ yr.}$$

$\therefore$ There is not enough time available

(2)

Tidal force: $F_{\text{tidal}} = \frac{GM_{\text{BH}}}{d^2} - \frac{GM_{\text{BH}}}{(d + \Gamma^*_d)^2}$

$$= \frac{GM_{\text{BH}}}{d^2} \left[ 1 - \left(1 + \frac{\Gamma^*_d}{d}\right)^{-2} \right]$$

For $\Gamma^*_d << d$ this can be approximated as $1 - 2\frac{\Gamma^*_d}{d} + \ldots$
\[ F_{\text{tidal}} = \frac{2GM_{BH}}{d^3} \]

Self-gravity force of the star:

\[ F_{\text{gy}} = \frac{GM_\star}{r_{\star}^2} \text{ (again force per unit mass)} \]

Equivalently, the critical distance is:

\[ d = \left( \frac{2M_{BH}}{M_\star} \right)^{\frac{1}{3}} \Gamma_{\star} \]

... with distortion @ smaller radii. As quoted ignoring the \(2^{1/3}\) numerical constant.

For the Galactic Center: \( M_{BH} = 4.5 \times 10^6 M_\odot \)

\[ d = \left( \frac{4.5 \times 10^6 M_\odot}{M_\odot} \right)^{\frac{1}{3}} \Gamma_{\odot} \]

\[ \approx 7 \times 10^8 \text{ cm} \]

\[ \Rightarrow \text{ tidal radius } d \approx 1.15 \times 10^3 \text{ cm.} \]

Schwarzschild radius \( R_s = \frac{2GM_{BH}}{c^2} = 1.3 \times 10^{12} \text{ cm.} \)

\[ \Rightarrow \text{ tidal radius is at } \approx 8.6 \text{ Schwarzschild radii.} \]