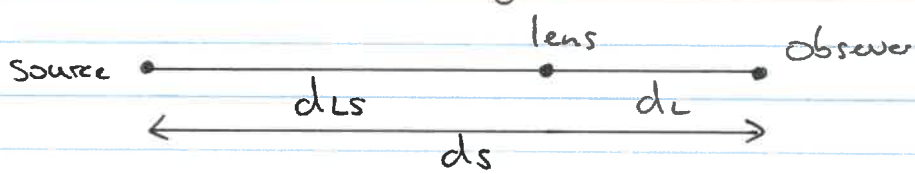
Problem Set #2: Solutions

(1) Formula for angular size of Einstein ring is:

$$\theta_E = \frac{2}{c}\sqrt{\frac{GM\, d_{LS}}{d_L\, d_S}}$$

... where the distances are defined via:



Let $d_L = x\, d_S$

$d_{LS} = (1-x)\, d_S$.

$$\Rightarrow \quad A = \pi d_L^2 \theta_E^2 = \frac{4\pi}{c^2}\, GM\, \frac{d_L\, d_{LS}}{d_S}$$

$$= \text{const} \times x(1-x).$$

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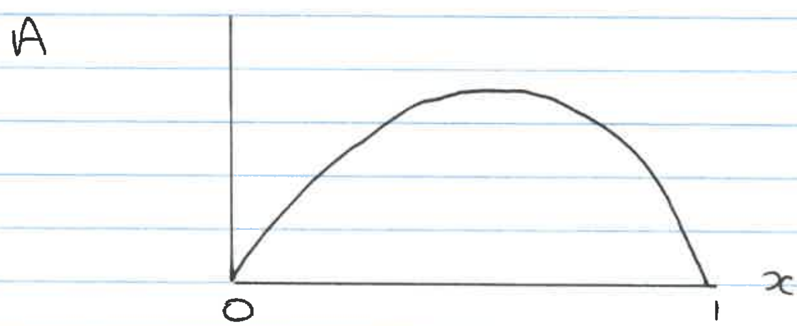

$$A = \text{const} \times (x - x^2)$$

$$\frac{dA}{dx} = \text{const}\,(1-2x) = 0$$

3

$$\Rightarrow \quad \underline{\underline{x_{max} = 1/2}}$$

(2) Mass function for $M > M_\odot$:

$$\xi(M) = \xi_0 M^{-2.35}.$$

Let mass of a white dwarf be M_{WD}.

Mass in remnant white dwarfs:

$$M = M_{WD} \times \text{number of white dwarfs}$$

$$= M_{WD} \int_{1M_\odot}^{8M_\odot} \xi(M)\, dM$$

$$= M_{WD}\, \xi_0 \left[\frac{M^{-1.35}}{-1.35} \right]_{1M_\odot}^{8M_\odot}$$

Total mass originally in stars:

$$M_{total} = \int_{1M_\odot}^{\infty} M \xi(M)\, dM = \xi_0 \left[\frac{M^{-0.35}}{-0.35} \right]_{1M_\odot}^{\infty}$$

Fraction locked up in WDs:

$$f = \frac{M}{M_{total}} = M_{WD} \times \frac{0.35}{1.35} \times \frac{\left[M^{-1.35} \right]_{1M_\odot}^{8M_\odot}}{\left[M^{-0.35} \right]_{1M_\odot}^{\infty}}$$

For $M_{WD} = 0.6\, M_\odot$

$$f = 0.6 \times \frac{0.35}{1.35} \times \left[\frac{8^{-1.35} - 1}{-1} \right]$$

(5)

$$\underline{\underline{f = 0.146}}$$

(3) Mass density near the Galactic center : $\rho = \rho_0 r^{-7/4}$

(a) Total mass within 2pc is $4.5 \times 10^6 M_\odot$

$$2\text{pc} = 6.172 \times 10^{18} \text{cm}$$
$$4.5 \times 10^6 M_\odot = 8.9505 \times 10^{39} \text{g.}$$

$$M = \int 4\pi r^2 \rho \, dr$$
$$= \int 4\pi \rho_0 r^{1/4} \, dr$$
$$= 4\pi \rho_0 \left[\frac{r^{5/4}}{5/4} \right]_0^{2\text{pc}}$$

$$\Rightarrow \quad \rho_0 = 2.89 \times 10^{15} \quad \text{in c.g.s. units.}$$

3 $$= 2.89 \times 10^{15} \ \text{g cm}^{-5/4}$$

2 (b) This is just $V = \sqrt{\frac{GM_{BH}}{r}}$

(c) Formula for dV/dt from the lectures is :

$$\frac{dV}{dt} = -\frac{4\pi G^2 M}{V^2} n m \ln \Lambda$$

... where M is the massive object, moving with velocity V with respect to less massive stars of mass m and number density n.

$$\ln \Lambda = \ln \left(\frac{b_{max}}{b_{min}} \right)$$ .. limits on impact parameter b.

$\therefore$ for the dynamical friction we have a scaling:

$$t_{friction} = \frac{V}{|dV/dt|} \propto \frac{V^3}{nm}$$

$$\left.\begin{array}{l} nm = \rho \propto r^{-7/4} \\ V \propto r^{-1/2} \end{array}\right\} \; t_{friction} \propto \frac{r^{-3/2}}{r^{-7/4}}$$

$$\propto r^{1/4}.$$

Now substitute numbers:

$$t_{friction} = \frac{V^3}{4\pi G^2 M \rho \ln \Lambda}$$

$$M = 10 M_\odot = 1.989 \times 10^{34} \, g$$

$$\rho = \rho_0 (1 \, pc)^{-7/4} = 1.272 \times 10^{-17} \, g\,cm^{-3}$$

$$V = 1.39 \times 10^{7} \, cm\,s^{-1}$$

$$\ln \Lambda = 10.$$

4 $\Rightarrow$ $\underline{\underline{t_{friction} \simeq 6.0 \times 10^{8} \, yr.}}$

Problem Set # 3: Solutions

(1)

$$\oint_S \underline{g} \cdot d\underline{A} = -4\pi G M_{in}$$

For a spherically symmetric mass distribution $\underline{g} = (-g_r, 0, 0)$ in polar co-ordinates. We have:

$$g_r \, . \, 4\pi r^2 = 4\pi G M_{in}$$

$$g_r = \frac{G M_{in}}{r^2}$$

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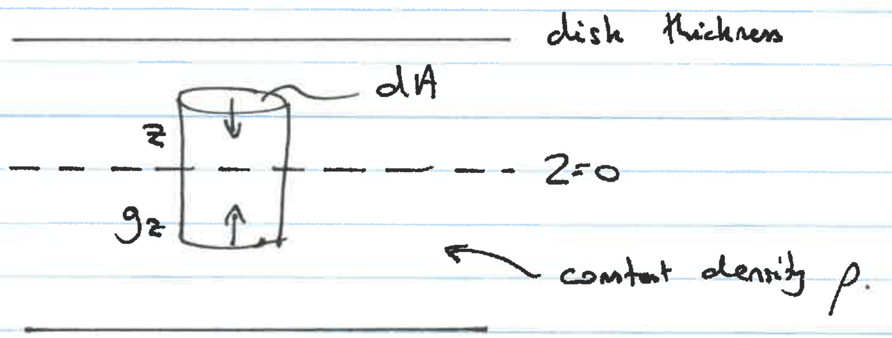
... as expected.

(2)



Consider a cylinder extending from $\pm z$ with cap area dA. By symmetry, $\underline{g} = 0$ except on end caps, where it equals g_z.

Applying formula above g_z (taking positive downwards):

$$2 g_z \, dA = 4\pi G \, . \, 2 z \, dA \, \rho.$$

$$\underline{\underline{g_z = 4\pi G \rho z}}$$

3

Since g_z is directed downwards, equation of motion is:

$$\frac{d^2 z}{dt^2} = -4\pi G \rho z$$

2 $$\frac{d^2 z}{dt^2} + kz = 0 \quad \text{with} \quad k = 4\pi G\rho.$$

General sol^n will ~~the~~ be sum of sin and cos terms.

e.g. trial sol^n

$$z = \sin(\omega t)$$
$$\dot{z} = \omega \cos(\omega t)$$
$$\ddot{z} = -\omega^2 \sin(\omega t)$$

$$\Rightarrow \quad -\omega^2 \sin(\omega t) + k \sin(\omega t) = 0$$
$$\omega = k^{1/2}.$$

3 Sol^n is $$\underline{\underline{z = A \sin\left[(4\pi G\rho)^{1/2} t\right] + B \cos\left[(4\pi G\rho)^{1/2} t\right]}}$$

For one oscillation: $\omega t = 2\pi$

$$\Rightarrow \quad P = \frac{2\pi}{(4\pi G\rho)^{1/2}} = \left(\frac{\pi}{G\rho}\right)^{1/2}$$

If $\rho = 0.1\ M_\odot\ pc^{-3} = 6.77 \times 10^{-24}\ g\,cm^{-3}$

2 $$\Rightarrow \quad P = 2.6 \times 10^{15}\ s$$

$$= \underline{\underline{8.35 \times 10^7\ yr}}$$

Problem Set # 4 - Solutions

(1) The Eddington limit is:

$$L_{Edd} = \frac{4\pi G c m_p}{\sigma_e} M$$

... where σ_e is the Thomson cross-section for electron-photon scattering.

If $L = \eta \dot{M} c^2$ with η = const.

$$\dot{M} = \frac{1}{\eta c^2} \frac{4\pi G c m_p}{\sigma_e} M$$

ie. $$\frac{dM}{dt} = kM \qquad \text{with} \quad k = \frac{4\pi G m_p}{\eta c \sigma_e}$$

Solving this eq^n: $$\int_{M_0}^{M} \frac{dM'}{M'} = \int_0^t k\, dt'$$

$$\ln\left[M/M_0\right] = kt$$

$$M = M_0 e^{kt} = M_0 e^{t/\tau}$$

... where $$\tau = k^{-1} = \frac{\eta c \sigma_e}{4\pi G m_p}$$

Numerically, for $\eta = 0.1$, $\tau = 1.4 \times 10^{15}$ s $= \underline{\underline{4.5 \times 10^7 \text{ yr}}}$

Note: strictly need to account for fact that radiated energy does not add to rest mass of black hole. In expression for τ this means $\eta \Rightarrow \eta/(1-\eta)$... but this correction is not too important.

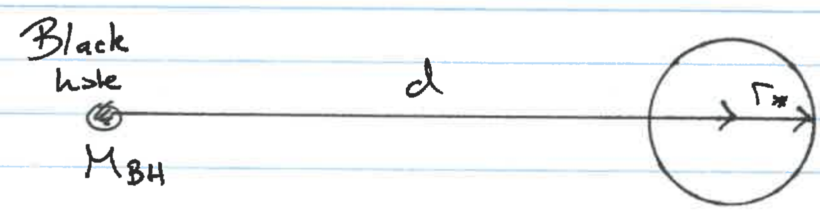
Time to grow from $10\,M_\odot \Rightarrow 10^9\,M_\odot$ by Eddington limited acc^n is then:

$$\ln\left[\frac{10^9}{10}\right] \times \tau = 18.4\,\tau$$

$$= 8.3 \times 10^8\ \text{yr}.$$

$\therefore$ There is <u>not</u> enough time available

(2)



Tidal force per unit mass (i.e. tidal acc^n):

$$F_{tidal} = \frac{GM_{BH}}{d^2} - \frac{GM_{BH}}{(d+r_*)^2}$$

$$= \frac{GM_{BH}}{d^2}\left[1 - \left(1 + \frac{r_*}{d}\right)^{-2}\right]$$

for $r_* \ll d$ this can be approximated as $1 - \frac{2r_*}{d} + \ldots$

$\Rightarrow$ $$F_{tidal} = \frac{2GM_{BH}}{d^3} r_*$$

Self-gravity force of the star:

$$F_{sg} = \frac{GM_*}{r_*^2}$$ (again force per unit mass)

Equating, the critical distance is:

$$d = \left(\frac{2M_{BH}}{M_*}\right)^{1/3} r_*$$

... with disruption @ smaller radii. As quoted ignoring the $2^{1/3}$ numerical constant.

For the Galactic Center : $M_{BH} \simeq 4.5 \times 10^6 M_\odot$

$$d \simeq \left(\frac{4.5 \times 10^6 M_\odot}{M_\odot}\right)^{1/3} r_\odot$$

7×10^{10} cm

$\Rightarrow$ tidal radius $d \simeq 1.15 \times 10^{13}$ cm.

Schwarzschild radius $R_s = \frac{2GM_{BH}}{c^2} = 1.3 \times 10^{12}$ cm.

$\Rightarrow$ Tidal radius is at $\simeq$ 8.6 Schwarzschild radii