Chapter 4

Derivation of Equations of Motion

This chapter will be divided into two main sections. The first section will start with Maxwell's equations and end with the equation for the laser field in a medium with a polarization $\mathcal{P}$. The second section will derive equations of motion for the density matrix of a closed two-level atomic system. The off-diagonal elements of the density matrix can be related to the polarization $\mathcal{P}$ of the atomic medium. The end result is a set of coupled equations for the propagation of the laser and the evolution of the atomic medium.

4.1 Field Equations

In the beginning (or shortly thereafter) there was light as described by Maxwell's equations,

$$\nabla \cdot \mathbf{D} = \rho_f, \tag{4.1a}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4.1b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{4.1c}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}, \tag{4.1d}$$

where, $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ and $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$ [29]. For the purpose of the discussion in this thesis we may take a special case of Maxwell's equations. To model the propagation of an intense laser field in a medium of two-level atoms we may assume that there is no free charge ($\rho_f = 0$) or current ($\mathbf{J}_f = 0$), and the net magnetization is zero ($\mathbf{M} = 0$). By placing the expression for $\mathbf{D}$ into Eq. (4.1a) we arrive at the following result:

$$\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = 0. \tag{4.2}$$

We are only interested in a polarization that is parallel to $\mathbf{E}$; consequently, we may write $\mathbf{P} = \varepsilon_0 \chi(t, \mathbf{r})\mathbf{E}$. This allows the simplification of Eq. (4.2) to

$$\varepsilon_0 \nabla \cdot \mathbf{E} + \varepsilon_0(\mathbf{E} \cdot \nabla \chi(t, \mathbf{r}) + \chi(t, \mathbf{r}) \nabla \cdot \mathbf{E}) \approx \varepsilon_0[1 + \chi(t, \mathbf{r})]\nabla \cdot \mathbf{E} = 0 \tag{4.3}$$

if $\chi(t, \mathbf{r})$ varies slowly over a wavelength, which implies $\nabla \cdot \mathbf{E} = 0$ because $\varepsilon_0[1 + \chi(t)] \neq 0$. This seemingly trivial point justifies the elimination of the $\nabla \cdot \mathbf{E}$ term that will be encountered momentarily.

The physics that we are concerned with involves linearly polarized laser light propagating in a medium of two-level atoms. For the two-level atom system the only degree of freedom for the polarization vector is along the axis defined by the polarization of the incoming laser field. Imagine the classical analog of a random distribution of fixed dielectric spheres. It is obvious that the induced dipole is in the direction of the driving field. In the case of real atoms, collisions may populate different magnetic states resulting in incoherent redistribution of light. This is inconsequential since we are only interested in the coherent propagation of the laser field in the forward direction and the loss of energy that is incoherently scattered into 4π is accounted for via an incoherent loss term in the equations of motion.

By taking the time derivative of Eq. (4.1d), and using Eq. (4.1c) along with the vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ the following wave equation is produced:

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}. \tag{4.4}$$

This wave equation can be simplified by considering propagation of the light in one direction (z), and using the assumption that the laser pulse envelope varies slowly in time and space compared to a period and wavelength of light. This approximation is called the Slowly Varying Envelope Approximation (SVEA). To apply the SVEA to Eq. (4.4) let the field $\mathbf{E}$ and the polarization $\mathbf{P}$ be written as an envelope function with an explicit time and space variation as follows:

$$\mathbf{E} = \frac{1}{2}(\varepsilon(x,y,z,t)e^{i(\omega_l t - \mathbf{K}_l \cdot \mathbf{z})} + \varepsilon^*(x,y,z,t)e^{-i(\omega_l t - \mathbf{K}_l \cdot \mathbf{z})})\hat{x} \tag{4.5a}$$

$$\mathbf{P} = \frac{1}{2}(\mathcal{P}(x,y,z,t)e^{i(\omega_l t - \mathbf{K}_l \cdot \mathbf{z})} + \mathcal{P}^*(x,y,z,t)e^{-i(\omega_l t - \mathbf{K}_l \cdot \mathbf{z})})\hat{x}, \tag{4.5b}$$

where ε and $\mathcal{P}$ are the envelope functions for the field and the polarization respectively, ω_l is the frequency of the laser and $\mathbf{K}_l$ is the free space wave number. One might imagine that the wave number for the laser will change once inside the medium and because of diffraction. All variations of the wave number from the free space wave number are contained inside the envelope function ε. This does not violate the SVEA because the magnitude of the index of refraction involved in this problem is small ($|n-1| \sim 10^{-5}$). Because we have picked the propagation direction to be the z direction we may replace $\mathbf{K}_l \cdot \mathbf{z}$ with $K_l z$.

By substituting Eq. (4.5a) and Eq. (4.5b) into Eq. (4.4) and dropping small terms of the form $\frac{\partial^2 \varepsilon}{\partial t^2}$, $\frac{\partial^2 \varepsilon}{\partial z^2}$, $\frac{\partial^2 \mathcal{P}}{\partial t^2}$ and $\frac{\partial \mathcal{P}}{\partial t}$, then collecting terms that oscillate as $e^{i(\omega_l t - \mathbf{K}_l \cdot \mathbf{z})}$ we find the slowly-varying Maxwell's equation

$$\frac{\partial \varepsilon}{\partial z} + \frac{1}{c}\frac{\partial \varepsilon}{\partial t} = -\frac{i\lambda}{4\pi}\nabla_\perp^2 \varepsilon - \frac{i\mu_0 \omega_l^2 \lambda}{4\pi}\mathcal{P}, \tag{4.6}$$

where λ is the wavelength of the laser light [38]. The justification for dropping the second derivative terms comes from the observation that $|\frac{\partial^2}{\partial t^2}| \ll |2i\omega_l \frac{\partial}{\partial t}|$ and $|\frac{\partial^2}{\partial z^2}| \ll |2iK_l \frac{\partial}{\partial z}|$. The $\frac{\partial \mathcal{P}}{\partial t}$ term is small and can be dropped if $\chi(t) \ll 1$. Only first derivatives are kept in the z and t coordinates thus the SVEA ignores backwards propagation.

The last manipulation of the field equation is motivated out of convenience. It is possible to combine the two partial derivatives into one by transforming to a co-moving frame defined as

$$t_c = t - \frac{z}{c} \tag{4.7a}$$

$$z_c = z. \tag{4.7b}$$

These transformations result in the following replacements:

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t_c} \tag{4.8a}$$

$$\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial z_c} \tag{4.8b}$$

which finally produce the co-moving slowly-varying Maxwell's equation

$$\frac{\partial \varepsilon(x,y,z_c,t_c)}{\partial z_c} = -\frac{i\lambda}{4\pi}\nabla_\perp^2 \varepsilon(x,y,z_c,t_c) - \frac{i\mu_0 \omega_l^2 \lambda}{4\pi}\mathcal{P}(x,y,z_c,t_c). \tag{4.9}$$

A transformation to a co-moving frame can be visualized as riding along on a section of the pulse and watching how the pulse envelope changes. A slightly different visualization of the same transformation is used when imagining the propagation of the laser in the computer code. Imagine a laser pulse envelope broken up into many individual pieces of infinitesimal width, like a histogram with very thin bars. We can imagine a single piece of this pulse propagating through a medium of two-level atoms. The polarization that this piece of the pulse “sees” is a direct result of all of the pieces of the pulse that have already passed. If, we were to stop time we would see the first piece of the pulse in an undisturbed medium with the piece behind the first in a medium effected by the first piece and so forth back to our piece of interest. This obvious picture is for the normal coordinates (z, t). If instead we send the first piece all the way through the medium without the second right behind we would be left with a modified medium. The second piece of the pulse may now be propagated through this modified medium leaving a medium that has now been modified by the first two pieces of the pulse. This process may be continued for each infinitesimal piece of the pulse. This second picture is of a co-moving frame.

From this point on it will be assumed that every equation is written in the co-moving frame making the subscript c redundant. Therefore, the subscript c will be dropped.

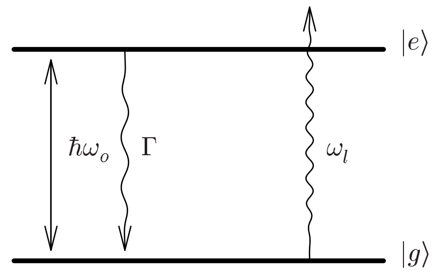

Figure 4.1: **Two-level atom:** The energy spacing between $|e\rangle$ and $|g\rangle$ is $\hbar\omega_o$. The natural decay rate is $\Gamma = 32.5\,\mathrm{MHz}$. A laser detuned to the blue is shown schematically on the right with a frequency ω_l.

4.2 Density matrix

In this section we will derive the equation of motion for the density matrix that describes a two-level atom in a field. Throughout this discussion it will be assumed that the wavelength (λ) of light is large when compared to the size of the atom. This is called the dipole approximation and will allow the addition of $-\wp \cdot \mathbf{E}$, where $\wp$ is the dipole moment of the atom, to the Hamiltonian for the atom. Many approximations will be glossed over in this chapter.

4.2.1 Bare closed two-level atom

The model under consideration here is an atom with only two levels. These levels will be referred to as $|g\rangle$ and $|e\rangle$ for ground and excited states with an energy separation of $\hbar\omega_o$ as shown in Fig. 4.1. The atom is driven by a classical field with frequency ω_l.

The Hamiltonian for a two-level atom in a classical field is as follows:

$$\mathbf{H} = \mathbf{H}_o + \mathbf{V} = \frac{1}{2}\hbar\omega_o \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 0 & \wp \\ \wp^* & 0 \end{pmatrix} E, \tag{4.10}$$

where $E = \mathbf{E} \cdot \hat{x}$. The equation of motion for the density matrix is

$$\dot{\boldsymbol{\rho}} = -\frac{i}{\hbar}\left[\mathbf{H}, \boldsymbol{\rho}\right], \tag{4.11}$$

where the density matrix is defined as

$$\boldsymbol{\rho} = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix} \tag{4.12}$$

because we have restricted the model to a closed system $\rho_{ee} + \rho_{gg} = 1$ which implies $\dot{\rho}_{ee} = -\dot{\rho}_{gg}$. We may choose $\wp = \wp^*$ because our atom only has two-levels which implies $\rho_{eg} = \rho_{ge}^*$. With these constraints, even though Eq. (4.11) generates six real equations we only need three of them to describe the physics. These three equations can be written as one real equation which contains the population information and one complex equation which contains the polarization information

$$\dot{\rho}_{ee} = -i\frac{\wp E}{\hbar}(\rho_{eg} - \rho_{ge}) - \Gamma\rho_{ee} \tag{4.13a}$$

$$\dot{\rho}_{eg} = -i\omega_o\rho_{eg} - i\frac{\wp E}{\hbar}(\rho_{ee} - \rho_{gg}) - \gamma\rho_{eg}, \tag{4.13b}$$

The two decay terms $-\Gamma\rho_{ee}$ and $-\gamma\rho_{eg}$ were added phenomenologically and did not come from Eq. (4.11). The decay term in Eq. (4.13a) is proportional to Γ which is the natural decay rate of the upper state population. The decay term in Eq. (4.13b) is proportional to $\gamma = \Gamma/2 + \gamma_c$ which is the dephasing rate, where γ_c is the dephasing due to collisions.

We may define a quantity called the population inversion as $d = \rho_{ee} - \rho_{gg}$ giving the following equations of motion for d and ρ_{eg}:

$$\dot{d} = -2i\frac{\wp E}{\hbar}(\rho_{eg} - \rho_{eg}^*) - \Gamma(d+1) \tag{4.14a}$$

$$\dot{\rho}_{eg} = -(\gamma + i\omega_o)\rho_{eg} - i\frac{\wp E}{\hbar}d. \tag{4.14b}$$

It should be pointed out that the value 1 in Eq. (4.14a) is actually $-d_o$ which is the initial condition for the population inversion. We have assumed that the initial population inversion is -1 which means that all of the atoms are in their ground state before the field is turned on.

The polarization in Eq. (4.9) is for a given point in space and time (x, y, z, t), but the atoms, which are governed by Eqs. (4.14a) and (4.14b), are moving through this point with some velocity distribution. To calculate the polarization at a given point we must use the concept of a convective derivative

$$\dot{\rho}_{eg} = \frac{d\rho_{eg}}{dt} = \frac{\partial\rho_{eg}}{\partial t} + v\frac{\partial\rho_{eg}}{\partial z}, \tag{4.15}$$

where v is the velocity in the z direction. This expression can be used in the co-moving frame, if $v \ll c$. The perpendicular velocity $v_\perp$ may be ignored since $v_\perp \tau_{\text{FWHM}} \ll \tilde{r}_{1/2}$ where τ_{FWHM} is the time duration of the pulse, and $\tilde{r}_{1/2}$ is a measure of the radius of the beam. The quantity $\dot{\rho}_{eg}$ is the change in ρ_{eg} as seen by an observer riding on an atom, where $\frac{\partial\rho_{eg}}{\partial t}$ is the change in ρ_{eg} as seen by an observer at a fixed point (x, y, z, t).

At this point it is convenient to remove the oscillations at the laser frequency from Eq. (4.14b) by placing the Eq. (4.5a) for the field into Eqs. (4.14a) and (4.14b) and by making the following two substitutions:

$$\rho_{eg} = \rho e^{-i(\omega_l t - K_l z)} \text{ and } \frac{\varepsilon\wp}{\hbar} = \Omega, \tag{4.16}$$

where Ω is the Rabi frequency. After simplification and discarding terms that oscillate as $e^{\pm 2i(\omega_l t - K_l z)}$, which is the Rotating Wave Approximation (RWA) [3], the following equations of motion are found:

$$\frac{\partial d(v)}{\partial t} = -i(\Omega\rho(v) - \Omega^*\rho^*(v)) - \Gamma(d(v)+1) \tag{4.17a}$$

$$\frac{\partial \rho(v)}{\partial t} = -(\gamma - i\Delta_o + iK_l v)\rho(v) - \frac{i}{2}\Omega^* d(v). \tag{4.17b}$$

Note that the v dependence is shown explicitly and $\Delta_o = \omega_l - \omega_o$ is the detuning of the laser from the atomic transition such that blue detuning is positive. Later the symbol Δ will be used which will include other terms than just the laser detuning.

It is now possible to write the full polarization $P = \mathbf{P}\cdot\hat{x}$ in terms of the density matrix as

$$P = \frac{1}{2}(\mathcal{P}e^{i(\omega_l t - K_l z)} + \mathcal{P}^* e^{-i(\omega_l t - K_l z)}) = N\int_{-\infty}^{\infty} \mathrm{Tr}[\wp\boldsymbol{\rho}(v)]W(v)dv, \tag{4.18}$$

where N is the number density and $W(v)$ is the normalized velocity distribution in the z direction. Using Eq. (4.5b), the polarization $\mathcal{P}$ which appears in Eq. (4.9) can be written as

$$\mathcal{P} = 2\wp N\int_{-\infty}^{\infty} \rho^*(v)W(v)dv. \tag{4.19}$$

By placing Eq. (4.19) into Eq. (4.9), the co-moving slowly-varying Maxwell's equation, and rewriting in terms of the Rabi frequency one arrives at the following equation of motion for the propagation of a laser through a medium of two-level atoms:

$$\frac{\partial\Omega}{\partial z} = -\frac{i\lambda}{4\pi}\nabla_\perp^2\Omega - \frac{3i}{4\pi}N\lambda^2\Gamma\int_{-\infty}^{\infty}\rho^*(v)W(v)dv. \tag{4.20}$$

The relationship

$$|\wp|^2 = \frac{3\pi\varepsilon_o\hbar c^3\Gamma}{\omega_o^3} \tag{4.21}$$

corresponding to a $\mathrm{J}_g = 0$ to $\mathrm{J}_e = 1$, transition was used along with the approximation $\omega_l/\omega_o \approx 1$ to simplify Eq. (4.20). By placing the velocity distribution $W(v)$ in Eq. (4.18) we have assumed that the velocity distribution of the ground state atoms is the same as the excited state atoms. This is true if the laser is detuned many Doppler widths from the atomic line, since there is no velocity selection in the excitation.

An alternative set of transformations to those used in Eq. (4.16) are

$$\rho_{eg} = \rho_\Phi e^{-i(\omega_l t - K_l z)} e^{-i\Phi} \text{ and } \frac{\varepsilon\wp}{\hbar} = |\Omega| e^{i\Phi}, \tag{4.22}$$

where the ρ is given the subscript Φ only to identify the use of a different transformation. These transformations produce the following equations of motion for $d(v)$ and $\rho_\Phi(v)$:

$$\frac{\partial d(v)}{\partial t} = -i|\Omega|(\rho_\Phi(v) - \rho_\Phi^*(v)) - \Gamma(d(v)+1) \tag{4.23a}$$

$$\frac{\partial \rho_\Phi(v)}{\partial t} = -(\gamma - i\Delta)\rho_\Phi(v) - \frac{i}{2}|\Omega| d(v), \tag{4.23b}$$

where $\Delta = \Delta_o + \dot{\Phi} - K_l v$. If the same expansion for the polarization that is used in Eq. (4.18) is kept, then the expression for $\mathcal{P}$ must be modified to included Φ in the following way:

$$\mathcal{P} = 2\wp N \int_{-\infty}^{\infty} \rho_\Phi^*(v) W(v) e^{-i\Phi} dv. \tag{4.24}$$

Both sets of transformations are valid and result in the same equations of motion for the laser field. In sub-section 4.2.2 density matrix equations in the dressed atom frame will be derived. It is more desirable to keep track of the phase of the polarization via Eq. (4.24) and define transformation coefficients that are real, rather than define complex transformation coefficients which carry the information of the phase of the polarization.

Often it is convenient to work with dimensionless quantities, as was shown in chapter 3 to map out the experimental parameter space. The longitudinal and radial directions $(z, \tilde{r})$ can be transformed into dimensionless parameters (ζ, r) via the following transformations:

$$z = \eta\zeta = \left[\frac{8\pi|\Delta_o|}{3N\lambda^2\Gamma}\right]\zeta \tag{4.25a}$$

$$\tilde{r} = \sigma_r r = \left[\sqrt{\frac{2|\Delta_o|}{3N\lambda\Gamma}}\right] r. \tag{4.25b}$$

These transformations allow Eq. (4.20) to be rewritten in the following form:

$$\frac{\partial\Omega}{\partial\zeta} = -i\nabla_\perp^2\Omega - 2i|\Delta_o| \int_{-\infty}^{\infty} \rho^*(v) W(v) dv. \tag{4.26}$$

or

$$\frac{\partial\Omega}{\partial\zeta} = -i\nabla_\perp^2\Omega - 2i|\Delta_o| \int_{-\infty}^{\infty} \rho_\Phi^*(v) W(v) e^{i\Phi} dv. \tag{4.27}$$

Three equations (4.26), (4.17a) and (4.17b) or (4.27), (4.23a) and (4.23b) along with an appropriate velocity distribution $W(v)$ contain all of the physics that will be discussed in this thesis. In principle all one needs is to solve these equations self consistently. In practice, this is quite difficult for numerous reasons. Analytical solutions can only be found for a few very special cases which will be shown in sections (5.1) and (5.4). Numerical solutions in principal are straightforward, but one quickly runs into limitations because computational time can become unreasonable. For example, if a calculation for a laser detuning of 200 GHz was done for a laser pulse of time length 6 ns it would take about 2 years to run on a DEC ALPHA 500 333 MHz computer. This excessive time arises from the fact that for a large detuning, the generalized Rabi frequency ($\mathcal{R}$) is large, and to produce accurate results one would have to take at least 10 time steps per $1/\mathcal{R}$.

4.2.2 Dressed atom

The dressed basis states may be written in terms of the bare atom states as

$$|1\rangle = b_2\,|e\rangle + b_1\,|g\rangle \tag{4.28a}$$

$$|2\rangle = b_1\,|e\rangle - b_2\,|g\rangle\,, \tag{4.28b}$$

where the coefficients b_1 and b_2 are defined in terms of the generalized Rabi frequency ($\mathcal{R} = \sqrt{|\Omega|^2 + \Delta^2}$) as

$$b_1 = \sqrt{\frac{\mathcal{R}+\Delta}{2\mathcal{R}}} \tag{4.29a}$$

$$b_2 = -\sqrt{\frac{\mathcal{R}-\Delta}{2\mathcal{R}}}, \tag{4.29b}$$

where $\Delta = \Delta_o + \dot{\Phi} - K_l v$ [38]. Using these transformations it is possible to write equations that transform the bare population inversion (d) and the off-diagonal matrix element (ρ_Φ) into the dressed population inversion $\mathcal{D}$ and the dressed off diagonal matrix element (σ),

$$\mathcal{D} = d(b_1^2 - b_2^2) - 2(\rho_\Phi + \rho_\Phi^*)b_1 b_2 \tag{4.30a}$$

$$\sigma = \rho_\Phi^* b_1^2 - \rho_\Phi b_2^2 + d b_1 b_2. \tag{4.30b}$$

The inverse transformations are

$$d = \mathcal{D}(b_1^2 - b_2^2) + 2(\sigma + \sigma^*)b_1 b_2 \tag{4.31a}$$

$$\rho_\Phi = \sigma^* b_1^2 - \sigma b_2^2 - \mathcal{D} b_1 b_2. \tag{4.31b}$$

The equations of motion for $\mathcal{D}$ and σ can be found by taking the derivatives of Eqs. (4.30a) and (4.30b) and using Eqs. (4.23a) and (4.23b). Note that $|\Omega|$ and Δ are functions of time which makes b_1 and b_2 functions of time. After some rather tedious algebra the following equations of motion are found:

$$\frac{\partial \mathcal{D}}{\partial t} = -\tilde{g}\mathcal{D} - f + 2q_+(\sigma + \sigma^*) \tag{4.32a}$$

$$\frac{\partial \sigma}{\partial t} = -(a + i\mathcal{R})\sigma + \chi + q_-\mathcal{D} + \mathrm{v}\sigma^* \tag{4.32b}$$

where

$$\tilde{g} = \frac{1}{\mathcal{R}^2}\left(\gamma|\Omega|^2 + \Gamma\Delta^2\right) \tag{4.33a}$$

$$f = \frac{\Gamma\Delta}{\mathcal{R}} \tag{4.33b}$$

$$q_+ = \frac{1}{2\mathcal{R}^2}\left(\Delta\frac{d}{dt}|\Omega| + |\Omega|\ddot{\Phi} - |\Omega|\Delta(\gamma - \Gamma)\right) \tag{4.33c}$$

$$q_- = \frac{1}{2\mathcal{R}^2}\left(-\Delta\frac{d}{dt}|\Omega| + |\Omega|\ddot{\Phi} - |\Omega|\Delta(\gamma - \Gamma)\right) \tag{4.33d}$$

$$a = \frac{1}{2\mathcal{R}^2}\left(\gamma\mathcal{R}^2 + \gamma\Delta^2 + \Gamma|\Omega|^2\right) \tag{4.33e}$$

$$\chi = \frac{\Gamma|\Omega|}{2\mathcal{R}} \tag{4.33f}$$

$$\mathrm{v} = \frac{|\Omega|^2}{2\mathcal{R}}(\gamma - \Gamma) \tag{4.33g}$$

and Φ is the phase of $\Omega = |\Omega|e^{i\Phi}$. Notice that all of the coefficients $\tilde{g}$ through v have a magnitude that is approximately γ. Solving equations (4.32a) and (4.32b) with coefficients (4.33a) through (4.33g) then transforming back to the bare atom frame via Eq. (4.31b) to calculate the polarization needed for the field equation (4.27) may seem tedious and wasteful. However, it will be shown in section 5.3 that this is the preferred method. The reason one can see immediately, since the coupling between Eqs. (4.32a) and (4.32b), (the density matrix equations in the dressed frame) is of the order γ whereas the coupling between Eqs. (4.23a) and (4.23b), (the density matrix equations in the bare atom frame) is of order $|\Omega|$ which may be arbitrarily large.