

Chapter 10

Resonant Manipulation of the Fractional Quantum Hall Effect

10.1 Resonant theory of a rapidly rotating Bose gas

There is a considerable amount of interest in the behavior of rotating Bose gases confined to an effective two-dimensional space. Beginning with the generation of a single vortex [109], experiments have progressively created increasingly dense vortex arrays [110, 111]. By driving the stirring frequency ever closer to the critical trap frequency, experiments are now creating scenarios where there exist a similar number of vortices as particles. One of the major goals is to push these systems well into the regime where the system is thought to display strong correlation effects.

Recently, Wilkin *et al.* [112] have shown that there exists a direct mapping between the two-dimensional Hamiltonian of a rotating Bose gas and of electrons subjected to a transverse magnetic field. The analog of the magnetic field is realized by the angular rotation and the Coulomb interaction is replaced by the two-particle scattering. The group later went on to show, through exact diagonalization studies of a small number of particles, that one could generate a Laughlin state (as well as many other novel FQHE type states) when there exists on the order of one vortex per particle [113]. This was also confirmed by Paredes *et al.* [114]. Unfortunately, if a state such as the Laughlin state were to be created in the laboratory, it would remain a major challenge to resolve it from the next excited state due to the extreme fineness of the gap. The size of the energy gap, however, is directly related to the strength of interatomic interactions.

A natural strategy for increasing the gap, and thus making the FQHE states more accessible to experiment, would be to exploit a mechanism, such as a Feshbach resonance, to enhance the interaction strength. To account for the full effects of the resonance, however, we cannot simply scale the mean-field energy, but must incorporate the entire resonant structure into the model. We must, therefore, include the process of molecular formation to the description of the rotating system.

By introducing a bound state, we not only modify the relative interaction strength, we also introduce a physical mechanism for generating pair correlations between particles (see Fig. 10.1). In the context of two-dimensional condensed matter systems, such a mechanism, which may arise from the long range nature of the Coulomb interaction [115], can have a dramatic effect on the properties of the ground state. In a resonant atomic gas, although the range of the potential is quite small, the effective scattering length may well exceed the inter-particle spacing so that even a small overlap between particles may lead to significant interaction effects. The closed channel bound state responsible for this enhancement, in effect, matches pairs of atoms within the gas resulting in a large increase in correlations.

A natural starting point for studying resonant effects in a rotating Bose gas is to look at the effect of the resonant interactions on the ground state wavefunction. By focusing on the most fundamental FQHE state, the Laughlin state, we will show that as the detuning approaches the Feshbach resonance, the Laughlin state transforms into a unique, strongly correlated state [116].

We begin by writing down an effective Hamiltonian, in second quantized form, for a resonant gas of Bosons of mass m rotating in two dimensions with stirring frequency Ω approaching the trapping frequency ω (i.e., $\omega - \Omega \rightarrow 0^+$):

$$\begin{aligned} \hat{H} = & \int d^2x \hat{\psi}_a^\dagger(\mathbf{x}) \left[\frac{-1}{2m} (\nabla - i\mathbf{A}(\mathbf{x}))^2 \right] \hat{\psi}_a(\mathbf{x}) \\ & + \frac{1}{2} \int d^2x' \int d^2x \hat{\psi}_a^\dagger(\mathbf{x}) \hat{\psi}_a^\dagger(\mathbf{x}') U(\mathbf{x}, \mathbf{x}') \hat{\psi}_a(\mathbf{x}') \hat{\psi}_a(\mathbf{x}) \end{aligned} \quad (10.1)$$

$$\begin{aligned}
& + \int d^2x \hat{\psi}_m^\dagger(\mathbf{x}) \left[\frac{-1}{4m} (\nabla - 2i\mathbf{A}(\mathbf{x}))^2 + \nu \right] \hat{\psi}_m(\mathbf{x}) \\
& + \frac{1}{2} \int d^2x' \int d^2x \left[\hat{\psi}_m^\dagger\left(\frac{\mathbf{x} + \mathbf{x}'}{2}\right) g(\mathbf{x}, \mathbf{x}') \hat{\psi}_a(\mathbf{x}) \hat{\psi}_a(\mathbf{x}') + \text{H.c.} \right].
\end{aligned}$$

Here $\hat{\psi}_{a,m}^\dagger(\mathbf{x})$, $\hat{\psi}_{a,m}(\mathbf{x})$ are the creation and destruction operators for atoms and molecules which satisfy the commutation relations $[\hat{\psi}_1(\mathbf{x}), \hat{\psi}_2^\dagger(\mathbf{x}')] = \delta^{(3)}(\mathbf{x}, \mathbf{x}') \delta_{1,2}$, where $1, 2 \in \{a, m\}$. We define the two-dimensional vector potential $\mathbf{A}(\mathbf{x}) = (m\omega y, -m\omega x)$, $U(\mathbf{x}, \mathbf{x}')$ is the two-particle background scattering potential, $g(\mathbf{x}, \mathbf{x}')$ is the resonant coupling between the open and closed channel potentials, and ν is the detuning of the open channel continuum from the level of the bound state in the closed channel.

From the work of Wilkin et al. [112], if we were to neglect resonant effects, we would expect the many-body ground state to be described by the Laughlin wavefunction [106]

$$\Psi_L(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) = \prod_{i < j} (z_i - z_j)^2 \prod_k \exp(-|z_k|^2/2), \quad (10.2)$$

where the products run over the indices $i, j, k = (1, 2, \dots, N)$ at position $z = x + iy$ for N particles (as noted earlier, we are focusing on the Laughlin wavefunction arising from a $1/2$ filling fraction). We will see that the inclusion of the resonant terms in Eq. (10.1) can significantly modify the form of Eq. (10.2) due to the growth of two-particle correlations.

10.2 Application of Chern-Simons theory

We approach this problem by an extension of Chern-Simons theory [108] which allows us to develop a mean-field theory for the rotating system that has removed the complications of the associated rotation. This is done by constructing a composite particle composed of the original particle and an artificially attached number of flux quanta. The composite particle is designed so that the attached flux quanta cancel the total rotation of the original system leaving a system of non-rotating, interacting composite particles. For the resonant system, the composite particles can formally be

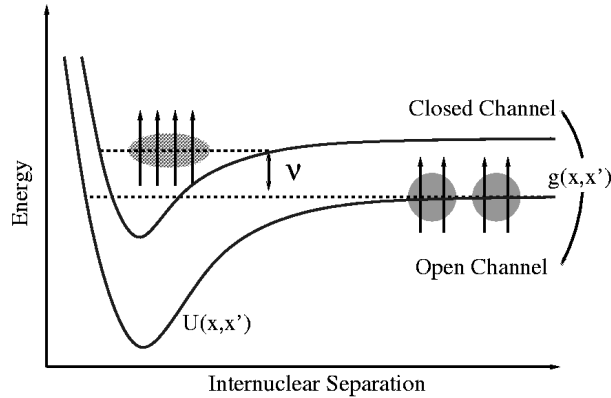


Figure 10.1: Pairs of composite atoms composed of single atoms and an associated number of angular momenta quanta (represented by the arrows) approach within an open channel potential of background value $U(\mathbf{x}, \mathbf{x}')$. They may form a composite molecule due to the presence of a closed channel bound state, at a detuning ν from the scattering continuum, which is coupled to the open channel with strength $g(\mathbf{x}, \mathbf{x}')$.

obtained by the following transformation:

$$\hat{\varphi}_{a,m}(\mathbf{x}) = \exp \left[-2i \int d^2x' \theta(\mathbf{x} - \mathbf{x}') (\rho_a(\mathbf{x}') + 2\rho_m(\mathbf{x}')) \right] \hat{\psi}_{a,m}(\mathbf{x}), \quad (10.3)$$

where $\rho_a(\mathbf{x}')$ and $\rho_m(\mathbf{x}')$ are the atomic and molecular spatial densities, respectively, and $\theta(\mathbf{x} - \mathbf{x}')$ is the topological phase [104].

In the composite picture, we replace the $\hat{\psi}$ operators in Eq. (10.1) with the corresponding composite operators $\hat{\varphi}$ and introduce the statistical Chern-Simons field $\mathbf{a}(\mathbf{x})$ through the gauge transformation

$$\mathbf{A}(\mathbf{x}) \rightarrow \mathbf{A}(\mathbf{x}) + \mathbf{a}(\mathbf{x}). \quad (10.4)$$

These modifications generate the Hamiltonian formulation of our composite atom/molecule theory. The composite picture can be shown to be equivalent to the single particle picture of Eq. (10.1).

We will now shift to a functional representation of the composite atom/molecule system to clarify the resonant modifications of the Chern-Simons theory and then return to the Hamiltonian formulation to derive the ground state wavefunction. Defining the

action within one temporal and two spatial dimensions

$$S = \int d^3x \sum_{\sigma=a,m} \varphi_{\sigma}^*(\mathbf{x}) i \partial_0 \varphi_{\sigma}(\mathbf{x}) - \int dx_0 \hat{H}, \quad (10.5)$$

we generate a Chern-Simons term which couples to the statistical vector field $a_{\mu}(\mathbf{x})$

$$S_{CS} = - \int d^3x \frac{1}{8\pi} \epsilon^{\mu\nu\lambda} a_{\mu}(\mathbf{x}) \partial_{\nu} a_{\lambda}(\mathbf{x}), \quad (10.6)$$

where the indices of μ, ν , and λ run over the three dimensions $(0, 1, 2)$ and a summation convention over repeated indices is invoked. We have also introduced the antisymmetric tensor $\epsilon^{012} = 1$. To simplify the following calculations we assume contact interactions of the form $U(\mathbf{x}, \mathbf{x}') = U\delta(\mathbf{x}, \mathbf{x}')$ and $g(\mathbf{x}, \mathbf{x}') = g\delta(\mathbf{x}, \mathbf{x}')$. Any complications arising from this replacement of the true potentials with contact potentials should be remedied as explained in Appendix A.

We next perform the lowest order variation of the action. Varying with respect to the zeroth component of the gauge field

$$\frac{\partial S}{\partial a_0} = 0, \quad (10.7)$$

reproduces the Chern-Simons condition

$$\nabla \times \mathbf{a}(\mathbf{x})|_z = -4\pi \left(|\varphi_a(\mathbf{x})|^2 + 2|\varphi_m(\mathbf{x})|^2 \right). \quad (10.8)$$

Equation (10.8) is a statement of Gauss's law for the statistical gauge field associating an even number of rotational flux quanta with each particle. This relation is simply a restatement of our choice of quasi-particle.

10.3 Calculation of the ground state wavefunction

Since we will be interested in the ground state properties of the atom/molecule system, let us assume that the fields $\varphi_a(\mathbf{x})$ and $\varphi_m(\mathbf{x})$ are uniform. By minimizing the action with respect to the atomic and molecular fields

$$\frac{\partial S}{\partial \varphi_a} = 0 \quad \text{and} \quad \frac{\partial S}{\partial \varphi_m} = 0, \quad (10.9)$$

we generate the following constraint equation for the molecules:

$$\varphi_m = \frac{g\varphi_a^2}{2[\nu + |\mathbf{A} + \mathbf{a}|^2/m]}. \quad (10.10)$$

Equation (10.10) allows us to eliminate the molecular field from the theory and arrive at a self-consistent relationship for the gauge field

$$|\mathbf{A} + \mathbf{a}|^2 = \left(U + \frac{g^2}{4(\nu + |\mathbf{A} + \mathbf{a}|^2/m)} \right) 2m|\varphi_a|^2. \quad (10.11)$$

Equation (10.11) is the usual result relating the gauge field to the background density only now it is dependent upon the detuning from the resonance.

We next switch back to the Hamiltonian form of our theory to derive the ground state wavefunction. After Fourier transforming the composite form of Eq. (10.1), by substitution of the field operators $\hat{\varphi}_a(\mathbf{x}) = \sum_{\mathbf{k}} \hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$ and $\hat{\varphi}_m(\mathbf{x}) = \sum_{\mathbf{k}} \hat{b}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$, we follow the usual Hartree-Fock-Bogoliubov (HFB) approach to construct a quadratic Hamiltonian which accounts for the lowest order pairing. As before, we assume contact interactions and now make the additional assumption that we may neglect the excited modes of the molecular field keeping only the lowest condensed mode $\hat{b}_{\mathbf{k}} = b_0$. This should remain a valid assumption so long as the molecular mode is not too greatly populated. The resulting Hamiltonian for the composite system can be written as:

$$H = H^0 + \sum_{\mathbf{k} \neq 0} A_{\mathbf{k}}^\dagger M_{\mathbf{k}} A_{\mathbf{k}}. \quad (10.12)$$

H^0 is composed of all terms of less than quadratic order in the operator $\hat{a}_{\mathbf{k}}$, we define a column vector $A_{\mathbf{k}} = (\hat{a}_{\mathbf{k}}, \hat{a}_{-\mathbf{k}}^\dagger)$, and $M_{\mathbf{k}}$ is the self-energy matrix. For our purposes we need only concern ourselves with the structure of the second term in Eq. (10.12). Here the self-energy matrix is expressed as

$$M_{\mathbf{k}} = \begin{pmatrix} E_{\mathbf{k}} & \Delta \\ \Delta^* & E_{-\mathbf{k}} \end{pmatrix} \quad (10.13)$$

with the additional definitions for the diagonal and off-diagonal terms

$$E_{\mathbf{k}} = E_{\mathbf{k}}^0 + U(|\varphi_a|^2 + n), \quad (10.14)$$

$$\Delta = U(\varphi_a^2 + p) + g\varphi_m. \quad (10.15)$$

Equations (10.14) and (10.15) are expressed in terms of the pairing-field $p = \sum_{\mathbf{k}'} \langle \hat{a}_{\mathbf{k}'} \hat{a}_{-\mathbf{k}'} \rangle$, the normal-field $n = \sum_{\mathbf{k}'} \langle \hat{a}_{\mathbf{k}'}^\dagger \hat{a}_{\mathbf{k}'} \rangle$, and $E_{\mathbf{k}}^0$ is the effective kinetic term which contains the contribution from the gauge field $A(\mathbf{x})$.

By rewriting Eq. (10.12) in terms of the quasi-particles

$$\hat{\alpha}_{\mathbf{k}} = \frac{1}{\sqrt{(E_{\mathbf{k}} + \omega_{\mathbf{k}})^2 + |\Delta|^2}} \left((E_{\mathbf{k}} + \omega_{\mathbf{k}}) \hat{a}_{\mathbf{k}} + \Delta \hat{a}_{-\mathbf{k}}^\dagger \right), \quad (10.16)$$

$$\hat{\alpha}_{-\mathbf{k}}^\dagger = \frac{1}{\sqrt{(E_{\mathbf{k}} + \omega_{\mathbf{k}})^2 + |\Delta|^2}} \left((E_{\mathbf{k}} + \omega_{\mathbf{k}}) \hat{a}_{-\mathbf{k}} + \Delta^\dagger \hat{a}_{\mathbf{k}} \right), \quad (10.17)$$

the result is a diagonal Hamiltonian

$$\mathcal{H} = \mathcal{H}^0 + \sum_{\mathbf{k} \neq 0} \omega_{\mathbf{k}} \hat{\alpha}_{\mathbf{k}}^\dagger \hat{\alpha}_{\mathbf{k}}, \quad (10.18)$$

where \mathcal{H}^0 contains the ground state contribution to the energy and the excitations are given by the spectrum of frequencies $\omega_{\mathbf{k}} = \sqrt{E_{\mathbf{k}}^2 - |\Delta|^2}$.

Since there are no quasi-particles present in the ground state $|gs\rangle$, which is what one would expect from an interacting bosonic system at $T = 0$, the ground state must satisfy the condition

$$\hat{\alpha}_{\mathbf{k}} |gs\rangle = 0. \quad (10.19)$$

Substitution of Eqs. (10.16) and (10.17) for the quasi-particle operators result in the relation

$$(E_{\mathbf{k}} + \omega_{\mathbf{k}}) \hat{a}_{\mathbf{k}} |gs\rangle = -\Delta \hat{a}_{-\mathbf{k}}^\dagger |gs\rangle. \quad (10.20)$$

Because $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^\dagger$ are canonically conjugate variables, there is no loss of generality in making the replacement $\hat{a}_{\mathbf{k}} \rightarrow \partial/\partial \hat{a}_{\mathbf{k}}^\dagger$ [117]. This converts Eq. (10.20) into a simple differential equation for the ground state with the solution:

$$|gs\rangle = \exp \left[\sum_{\mathbf{k}} \frac{-\Delta}{E_{\mathbf{k}} + \omega_{\mathbf{k}}} \hat{a}_{-\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}^\dagger \right] |0\rangle. \quad (10.21)$$

To derive the many-body wavefunction, we must now move from second to first quantization. We follow the method of reference [118] in what follows. The relationship which links the second quantized ground state $|gs\rangle$ with the first quantized wavefunction Ψ_{CB} can be written for an even number of noncondensed particles $2N$ as

$$\Psi_{CB}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{2N}) = \langle 0 | \hat{\varphi}_a(\mathbf{x}_{2N}) \dots \hat{\varphi}_a(\mathbf{x}_2) \hat{\varphi}_a(\mathbf{x}_1) | gs \rangle, \quad (10.22)$$

where it should be noted that Ψ_{CB} is the full-many body wavefunction for the composite Bose particles. If we are able to assume that $E_{\mathbf{k}} + \omega_{\mathbf{k}} \gg \Delta$, an assumption which will remain valid as long as we are not too close to resonance, we may truncate the power expansion of the exponent in Eq. (10.21). This results in the composite boson wavefunction

$$\Psi_{CB}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{2N}) = \mathcal{S}(\psi_{12}\psi_{34}\dots\psi_{(2N-1)2N}), \quad (10.23)$$

comprised of a symmetrized product \mathcal{S} of paired wavefunctions

$$\psi_{ij} = \sum_{\mathbf{k}} \frac{-\Delta}{E_{\mathbf{k}} + \omega_{\mathbf{k}}} e^{i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)}. \quad (10.24)$$

If we were dealing with a system of fermions, Eq. (10.23) would be antisymmetrized and would result in a Pfaffian wavefunction [119]. Here, because of the statistics of the particles, we generate a bosonic analogue to this result. The many-body wavefunction for the bare particles can now be extracted from the composite wavefunction (see reference [118]) resulting in:

$$\Psi_{MB} = \Psi_{CB}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{2N}) \times \Psi_L(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{2N}), \quad (10.25)$$

which is a product of the composite particle wavefunction of Eq. (10.23) and the Laughlin wavefunction of Eq. (10.2).

10.4 Discussion of resonant effects

Equation (10.25) is the final result for the ground state wavefunction of the resonant rotating Bose system. This result has important consequences for the generation

of a Laughlin state within a resonant atomic gas. It would imply that since a Feshbach resonance necessarily increases two-particle correlations between composite particles, resonantly increasing the interactions will eventually result in a modification to the ground state wavefunction. As is clear from the form of Eq. (10.23), for large detuning from the resonance, corresponding to small pairing and molecular field, the many-body wavefunction reduces to the Laughlin wavefunction. As one moves nearer to the resonance, however, the off-diagonal part of the self-energy matrix, Δ , grows. This results in an increasing modification of the many-body wavefunction from the Laughlin wavefunction. The ability to tune a Feshbach resonance, therefore, allows for the direct study of this crossover from a Laughlin wavefunction to a paired wavefunction.

A similar, yet distinct, Moore-Read type wavefunction, as in Eq. (10.25), was found for electronic FQHE systems [120]. This has been used to explain the previously unresolved, even denominator filling fractions which result from a pairing instability, such as the observed incompressibility of the $5/2$ filling. In this case, a straightforward generalization of the Laughlin wavefunction would result in a symmetric wavefunction, violating the asymmetry of the fermions. However, the generation of an antisymmetric Pfaffian wavefunction which multiplies the generalized Laughlin state allows the overall ground state to be correctly antisymmetrized. For the bosonic system we have treated, the overall wavefunction must remain symmetric, so the corresponding paired wavefunction is symmetric in comparison to the antisymmetric Pfaffian wavefunction.

In conclusion, the introduction of Feshbach resonance interactions to the problem of a rotating Bose gas forces one to account for the effects of molecular formation. The growth of pair correlations among the composite particles leads to a modification of the nonresonant ground state for a rapidly rotating Bose gas. The new ground state wavefunction, which is generated by the Feshbach resonance, exists as a strongly correlated state unique to trapped Bose gases. Currently, the major challenge in realizing the simplest of FQHE states, the Laughlin state, is the need to reach extremely low

temperatures to resolve the ground state from the excited states. Since the size of this energy gap is directly related to the strength of interatomic interactions, by increasing the interactions, it would seem natural that one could increase the energy gap, making the system more accessible to experiment. The results of this paper would imply that any attempt to increase the gap with a Feshbach resonance would result in modifying the ground state from the desired Laughlin state. Unfortunately, at the level of the present calculation, we are unable to comment on the resonant behavior of the gap so defer this discussion for now. These results, nonetheless, reveal the exciting possibility of directly studying the crossover transition between a bosonic Moore-Read type state and the Laughlin state.

These conclusions have several other important implications for the resonant production of FQHE states within atomic gases. For instance, many of the observable properties of the gas may be modified, such as the density profile for both atoms and molecules and the nature of collective excitations. It should also be noted that the crossover transition we have discussed is only a part of a much more general crossover theory made accessible by the tunability of a Feshbach resonance. Although the methods presented here are invalid close to the resonance, one could imagine extending these ideas to describe the resonant system as one passes from a gas of interacting rotating atoms, through the resonance, to a system of tightly bound, rotating molecules. This would be the analog for the rotating system of the crossover problem we have discussed at such length within this thesis.