Appendix D

Matsubara frequency summations

In finite-temperature field theories we often encounter Matsubara frequency summations over products of free Green's functions. In this appendix we will list some of the more common summations and detail how they are evaluated. A discussion of evaluating more complex summations than are presented here can be found in reference [87].

D.1 Common summations

In this thesis we have made use of the following useful relations:

$$\frac{1}{\beta}\sum_{i\omega_p^n} G_0(\mathbf{p}, i\omega_p^n) = n_f(\xi_{\mathbf{p}}), \tag{D.1}$$

$$-\frac{1}{\beta}\sum_{i\nu_p^n} D_0(\mathbf{p}, i\nu_p^n) = n_b(\xi_{\mathbf{p}}), \tag{D.2}$$

$$-\frac{1}{\beta}\sum_{i\omega_p^n} G_0(\mathbf{p}, i\omega_p^n) G_0(\mathbf{q}, i\omega_q^n - i\omega_p^n) = \frac{1 - n_f(\xi_{\mathbf{p}}) - n_f(\xi_{\mathbf{q}})}{i\omega_q^n - \xi_{\mathbf{p}} - \xi_{\mathbf{q}}}. \tag{D.3}$$

Equations (D.1) and (D.2) result in Fermi and Bose distributions defined, respectively, as:

$$n_f(\xi_{\mathbf{p}}) = \frac{1}{e^{\beta\xi_{\mathbf{p}}} - 1} \quad \text{and} \quad n_b(\xi_{\mathbf{p}}) = \frac{1}{e^{\beta\xi_{\mathbf{p}}} + 1}, \tag{D.4}$$

whereas Eq. (D.3) was encountered in our discussion of the pair susceptibility χ.

D.2 Evaluation of Matsubara summations

We begin our discussion of evaluating Matsubara series with the simple case of Eq. (D.1). If we explicitly write the free Fermion Green's function, the sum we wish to evaluate is:

$$S = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \frac{1}{i\omega_p^n - \xi_p}, \tag{D.5}$$

where the summation runs over integer values for the odd Matsubara frequencies $\omega_p^n = (2n+1)\pi/\beta$. Let us write this in the representative form

$$S = -\frac{1}{\beta} \sum_n f(i\omega_p^n), \tag{D.6}$$

where $f(i\omega_p^n)$ is defined through Eq. (D.5). To evaluate the sum, we transform the summation into an equivalent contour integration. Let us look at the following integral:

$$I = \lim_{R\to\infty} \int \frac{dz}{2\pi i} f(z) n_f(z). \tag{D.7}$$

We now chose the contour to be a circle of radius $R \to \infty$ and impose that the function $n_f(z)$ generates poles at all points $z_n = (2n+1)i\pi/\beta$. An appropriate function would be:

$$n_f(z) = \frac{1}{e^{\beta z} + 1}, \tag{D.8}$$

which has a residue of $-1/\beta$ at each pole z_n. The contour and these periodic poles are illustrated in Fig. D.1. The function $f(z)$ is defined as:

$$f(z) = \frac{1}{z - \xi_p}, \tag{D.9}$$

which contributes an extra pole at $z = \xi_p$. To clarify these results, we list all the poles and residues:

$$z_n = (2n+1)i\pi/\beta, \qquad R_n = -\frac{1}{\beta} \frac{1}{(2n+1)i\pi/\beta - \xi_p} \tag{D.10}$$

$$z_1 = \xi_p, \qquad R_1 = n_f(\xi). \tag{D.11}$$

According to Cauchy's theorem [126], Eq. (D.7) may be written as a sum of residues

$$I = -\frac{1}{\beta}\sum_n \frac{1}{(2n+1)i\pi/\beta - \xi_p} + n_f(\xi_p). \tag{D.12}$$

In the limit that we take the contour $R \to \infty$, assuming Jordan's lemma is satisfied [126], the integral $I \to 0$. We, therefore, have the result:

$$\frac{1}{\beta}\sum_n \frac{1}{(2n+1)i\pi/\beta - \xi_p} = n_f(\xi_p), \tag{D.13}$$

which may be written as

$$\frac{1}{\beta}\sum_{n=-\infty}^{\infty} \frac{1}{i\omega_p^n - \xi_p} = n_f(\xi_p). \tag{D.14}$$

This final result proves Eq. (D.1).

For bosons, the method is similar except the contour integral should be chosen to have poles at the even values $z_n = 2ni\pi/\beta$. It is therefore appropriate to replace Eq. (D.7) with:

$$I = \lim_{R\to\infty}\int \frac{dz}{2\pi i} f(z) n_b(z), \tag{D.15}$$

where the function $n_b(z)$ is given by

$$n_b(z) = \frac{1}{e^{\beta z} - 1}, \tag{D.16}$$

which now has a residue of $1/\beta$ at each pole z_n.

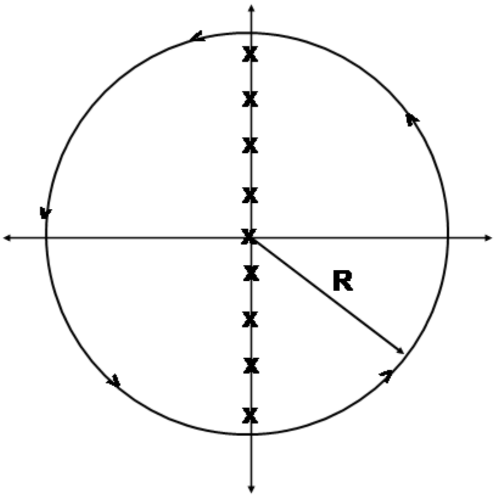

Figure D.1: A typical contour for Matsubara summation integrals. The x's represent evenly spaced poles along the vertical axis at points z_n and the contour radius is taken in the limit $R \to \infty$.