Appendix C

Functional Integrals and Grassmann Variables

Several useful integrals are often encountered when working with path integrals, the most ubiquitous being Gaussian integrals. We begin by discussing the evaluation of these integrals for the case of Bosons, where we may work with classical fields, and then proceed to discuss fermions and Grassmann variables [86, 125].

C.1 Bosons

For complex fields it is often useful to first decompose the fields into real and imaginary parts. For the bosonic fields x, x^* we may expand them as:

$$x = \frac{x_1 + ix_2}{\sqrt{2}}, \qquad x^* = \frac{x_1 - ix_2}{\sqrt{2}}. \tag{C.1}$$

Before solving for the basic Gaussian functional integral, it is useful to first look at the evaluation of a Gaussian integral of the complex variables x, x^*:

$$\int dx^* dx e^{-x^* a x} = \int dx_1 dx_2 e^{-\frac{1}{2}ax_1^2} e^{-\frac{1}{2}ax_2^2} = \left(\int dx e^{-\frac{1}{2}ax^2} \right)^2 = \frac{2\pi}{a}. \tag{C.2}$$

We have made use of Eq. (C.1) in the second step of Eq. (C.2). Equation (C.2) allows us to evaluate the Gaussian functional integral of the complex fields x, x^* acting on the matrix A with eigenvalues a_i:

$$\int dx^* dx \exp(-x^* A x) \quad = \quad \left(\prod_{i,j} \int dx_i^* dx_j \right) \exp(-\sum_{i,j} x_i A_{i,j} x_j) \tag{C.3}$$

$$
\begin{aligned}
&= \left(\prod_i \int dx_i^* dx_i\right) \exp(-\sum_i x_i^* a_i x_i) \\
&= \prod_i \frac{2\pi}{a_i} = \frac{(2\pi)^n}{\det A}.
\end{aligned}
$$

C.2 Fermions and Grassmann variables

Fermions require the introduction of Grassmann variables in order to satisfy the correct particle statistics. Grassmann variables are defined such that the exchange of any two variables is antisymmetric:

$$x\eta = -\eta x. \tag{C.4}$$

What's more, the product of more than one Grassmann variable yields zero:

$$x^2 = 0. \tag{C.5}$$

This definition may yield an ambiguity in sign when integrating these variables so we must choose a convention. Here we impose the convention $\int dx^* dx (xx^*) = 1$. Let's first look at how a simple Gaussian integral over Grassmann variables is modified:

$$\int dx^* dx \exp(-x^* ax) = \int dx^* dx(1 - x^* ax) = \int dx^* dx(1 + xx^* a) = a. \tag{C.6}$$

This should be contrasted to the result of Eq. (C.2) for a standard, classical variable. With this information we may evaluate the Gaussian functional integral for fermions:

$$
\begin{aligned}
\int dx dx^* \exp(-xAx) &= \left(\prod_{i,j} \int dx_i^* dx_j\right) \exp(-\sum_{i,j} x_i A_{i,j} x_j) \qquad \text{(C.7)} \\
&= \left(\prod_i \int dx_i^* dx_i\right) \exp(-\sum_i x_i^* a_i x_i) \\
&= \prod_i a_i = \det A.
\end{aligned}
$$

Note that for bosons the result was $(2\pi)^n / \det A$. Another useful integral that is often encountered is:

$$\frac{\int dx^* dx \exp(-x^* Ax + \eta^* x + x^* \eta)}{\int dx^* dx \exp(-x^* Ax)} = \exp(\eta^* A^{-1} \eta), \tag{C.8}$$

where

$$\eta^* x = \textstyle\sum_i \eta_i^* x_i, \qquad x^* \eta = \textstyle\sum_i x_i^* \eta_i. \tag{C.9}$$

The variables η_i, η_i^* anticommute with each other and x_i, x_i^*. This can be proved by a shift transformation

$$x \rightarrow x + \tilde{\eta}, \qquad x^* \rightarrow x^* + \tilde{\eta}^*, \tag{C.10}$$

which will cancel the linear terms in the exponent of Eq. (C.8). When applied in practice, this shift transformation is often known as a Hubbard-Stratonovich transformation.