

Appendix B

Bogoliubov Transformation

Very few many-body problems allow for an exact solution. Among the few that do, one of the more commonly encountered forms are Hamiltonians which are quadratic in creation and annihilation operators. In this appendix we will sketch the general method of solving these quadratic Hamiltonians for the case of both bosons and fermions. A much more thorough treatment can be found in various books such as that of reference [54].

B.1 Hermitian quadratic Hamiltonian

A general, quadratic Hamiltonian of this sort may be written in the form

$$H = \sum_{ij} A_{ij} a_i^\dagger a_j + \frac{1}{2} \sum_{ij} (B_{ij} a_i^\dagger a_j^\dagger + B_{ij}^* a_j a_i), \quad (\text{B.1})$$

where the following conditions are satisfied

$$A = A^\dagger, \quad B^\top = -\varepsilon B. \quad (\text{B.2})$$

The variable ε is defined such that $\varepsilon = 1$ for fermions and $\varepsilon = -1$ for bosons. Equation (B.1) may be written in matrix form as

$$H = \frac{1}{2} \alpha^\dagger M \alpha + \frac{\varepsilon}{2} \text{tr} A, \quad (\text{B.3})$$

by introducing the row and column vectors

$$\alpha = \begin{pmatrix} a \\ a^\dagger \end{pmatrix}, \quad \alpha^\dagger = (a^\dagger a), \quad (\text{B.4})$$

and defining the Matrix

$$M = \begin{pmatrix} A & B \\ -\varepsilon B^* & -\varepsilon A^* \end{pmatrix}. \quad (\text{B.5})$$

B.2 Unitary canonical transformation

We define the matrix T which moves us from the basis α to a diagonal basis β

$$\beta = T\alpha. \quad (\text{B.6})$$

We would like this transformation to be canonical, meaning that the operators in the new basis satisfy the same commutation relations as the original operators. This can be expressed by the matrix η defined as:

$$\eta = \begin{pmatrix} [a, a^\dagger]_\varepsilon & [a, a]_\varepsilon \\ [a^\dagger, a^\dagger]_\varepsilon & [a^\dagger, a]_\varepsilon \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}. \quad (\text{B.7})$$

In order for a transformation T to remain canonical, it must satisfy the following criteria:

$$T\eta T^\dagger \eta = 1 \quad T^* = \gamma T \gamma, \quad (\text{B.8})$$

where we have introduced the matrix

$$\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (\text{B.9})$$

It can clearly be seen that a canonical transformation of Eq. (B.3) results in the diagonal Hamiltonian

$$H = \frac{1}{2} \beta^\dagger \eta T \eta M T^{-1} \beta + \frac{\epsilon}{2} \text{tr} A, \quad (\text{B.10})$$

where $T \eta M T^{-1}$ is a diagonal matrix.

B.3 Basis transformations for bosons and fermions

A canonical basis transformation, known as the Bogoliubov transformation, that brings Eq. (B.3) into diagonal form is given by

$$\begin{pmatrix} b_\nu \\ b_\nu^\dagger \end{pmatrix} = \begin{pmatrix} \cosh \omega_\nu & e^{i\theta_\nu} \sinh \omega_\nu \\ e^{-i\theta_\nu} \sinh \omega_\nu & \cosh \omega_\nu \end{pmatrix} \begin{pmatrix} a_\nu \\ a_\nu^\dagger \end{pmatrix}, \quad (\text{B.11})$$

for a single component system of bosons and by

$$\begin{pmatrix} b_{\uparrow\nu} \\ b_{\downarrow\nu} \\ b_{\uparrow\nu}^\dagger \\ b_{\downarrow\nu}^\dagger \end{pmatrix} = \begin{pmatrix} \cos \omega_\nu & 0 & 0 & -ie^{i\theta_\nu} \sin \omega_\nu \\ 0 & \cos \omega_\nu & ie^{i\theta_\nu} \sin \omega_\nu & 0 \\ 0 & ie^{-i\theta_\nu} \sin \omega_\nu & \cos \omega_\nu & 0 \\ -ie^{-i\theta_\nu} \sin \omega_\nu & 0 & 0 & \cos \omega_\nu \end{pmatrix} \begin{pmatrix} a_{\uparrow\nu} \\ a_{\downarrow\nu} \\ a_{\uparrow\nu}^\dagger \\ a_{\downarrow\nu}^\dagger \end{pmatrix}, \quad (\text{B.12})$$

for a two-spin system of fermions. These transformations, of course, can be generalized to systems of higher degrees of freedom.