# Appendix A

## Phase-contrast imaging

Following is a more detailed derivation of phase contrast imaging, specializing to our specific case. The only assumptions made previously are qualified here; that the phase shift from a condensate is small, and that the probe beam size is much larger than the condensate. Due to the cylindrical symmetry along the probing direction (aside from a constant scaling factor) the analysis can be done in only one dimension, perpendicular to the direction of beam propagation.

I begin by writing down the electric field for a gaussian beam (of size $\sigma = 1/\sqrt{2a}$) propagating along the $\hat{z}$ direction with angular frequency $\omega$ and spatial phase profile $\phi(x) = \phi_0 \exp(-bx^2)$;

$$\begin{aligned} E(\vec{x}, \vec{z}) &= E_0 e^{-ax^2} e^{i\omega t + i\phi(x)} e^{-i\vec{k}\cdot\vec{z}} \\ &\simeq E_0 e^{-ax^2} (1 + i\phi(x) - \phi^2(x)/2) \end{aligned} \tag{A.1}$$

where the approximation is valid for small $\phi_0$, and the time and $\vec{z}$ terms can be dropped. This is the field just after the atom cloud. The field at the focal point (P1) of the first imaging lens (Fig. 4.10) is the Fourier transform of the field at the condensate, evaluated at $k = x/\lambda f_1$;

$$\begin{aligned} E(x, P1) &= \int_{-\infty}^{\infty} E_0 e^{-ax^2} \left(1 + i\phi(x) - \phi^2(x)/2\right) e^{-2\pi i k x} dx \\ &= e^{-\frac{\pi^2}{a\lambda^2 f_1^2}x^2} \left( \frac{1}{\sqrt{a}} + \frac{i\phi_0}{\sqrt{a+b}} e^{-\frac{\pi^2}{b\lambda^2 f_1^2}x^2} - \frac{\phi_0^2}{2\sqrt{a+2b}} e^{-\frac{\pi^2}{2b\lambda^2 f_1^2}x^2} \right) \end{aligned} \tag{A.2}$$

In this equation the coefficient to $x^2$ in the exponent is the size of the beam/phase profile in the focal plane of the first lens. The factors in front are also dropped for convenience, but can be easily replaced at the end by requiring the intensity to be normalized.

This is the point where the phase mask is inserted. The phase mask applies a spatially dependent shift of the following form

$$\theta(x) \begin{cases} \theta & \text{if } -c < x < c \\ 0 & \text{otherwise} \end{cases} \tag{A.3}$$

where $2c$ is the diameter of the magnesium fluoride dot responsible for the phase shift.

Again the Fourier transform is taken and evaluated at $k = x/\lambda f_2$ for the second imaging lens which focuses onto the CCD array. The transform is broken up into two parts due to the discontinuity in the phase dot;

$$\begin{aligned} E(x, P2) &= 2\int_0^c e^{i\theta} E(x, P1) + 2\int_c^\infty E(x, P1) \\ &= e^{-\frac{f_1^2}{f_2^2}ax^2}\Big((e^{i\theta}-1)A+1\Big) + i\phi_0 e^{-\frac{f_1^2}{f_2^2}(a+b)x^2}\Big((e^{i\theta}-1)B+1\Big) \\ &- \frac{\phi_0^2}{2} e^{-\frac{f_1^2}{f_2^2}(a+2b)x^2}\Big((e^{i\theta}-1)C+1\Big). \end{aligned} \tag{A.4}$$

The coefficients in the exponents are now the size of the beam/phase profile in the image plane, properly scaled by the magnification. $A$, $B$ and $C$ are error functions from the integrals in Eq. A.4;

$$\begin{aligned} A &\equiv \text{erf}\left(\frac{\pi c}{\lambda f_1 \sqrt{a}}\right) \\ B &\equiv \text{erf}\left(\frac{\pi c}{\lambda f_1 \sqrt{a+b}}\right) \\ C &\equiv \text{erf}\left(\frac{\pi c}{\lambda f_1 \sqrt{a+2b}}\right) \end{aligned} \tag{A.5}$$

where the argument of the error functions is the ratio of the size of the phase dot to the beam (in $A$) or phase profile (in $B$) in the plane P1.

To find the intensity at the CCD array, the field from Eq. A.4 is squared and time-averaged to give

$$\begin{aligned} I(x) &= 2I_0 e^{-\frac{f_1^2}{f_2^2} a x^2} \Big( (1 - \cos\theta) A^2 + (\cos\theta - 1) A + 1/2 \Big) \\ &+ 2I_0 \phi_0 e^{-\frac{f_1^2}{f_2^2} b x^2} (A - B) \sin\theta. \end{aligned} \tag{A.6}$$

This shows that the signal is proportional to the phase shift, and that the spatial distribution is recreated on the CCD. The best signal is obtained when $\theta = \pi/2$ and $A = 1$, $B = 0$. The first condition, $A = 1$, requires that the probe beam waist at the plane P1 is much smaller than the phase dot, and $B = 0$ requires that the diffracted wave from the condensate be much larger than the phase dot at P1. With these conditions the signal is

$$I(x) = I_0(1 + 2\phi(x)). \tag{A.7}$$