

**Quantum State Engineering and Information  
Processing with Trapped Ions**

by

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Quantum State Engineering and Information Processing with Trapped Ions

Thesis directed by Dr. David Wineland

A single  ${}^9\text{Be}^+$  ion confined in an rf (Paul) trap may be used to realize two of the simplest quantum systems: the two-level system and the harmonic oscillator. The two-level system is comprised of two, ground-state hyperfine electronic levels. The trapping potential is harmonic, to a high degree of approximation, and so the ion's motion is that of a three-dimensional harmonic oscillator.

By coupling the ion's motional and electronic degrees of freedom, we can engineer entanglement between these systems. This allows us to study quantum mechanics, with all its peculiarities, in a well-controlled environment. For example, we can study the interactions of superposition states with the ion's environment, leading to a destruction of quantum superpositions. Furthermore, this system, when scaled up to several ions, may allow us to construct a simple "quantum computer," which promises exponential speed-up over any possible classical computer for some computational problems. Towards this goal, we have cooled two, trapped ions to their ground state of collective motion and have entangled their electronic degrees of freedom by using their joint motion to transfer entanglement between them.

## Dedication

To my parents, Basil and Lena King, who first taught me to look, and to wonder...

## Acknowledgements

*For my part, I know nothing with any certainty, but the sight of the stars makes me dream...* — V. Van Gogh

This thesis represents the culmination of a very lengthy journey: one which I have not — and could not have — travelled on my own. So I would like to take this opportunity to thank those who have gotten me to this point. I cannot separate, in my mind or in my memories, the research described in this thesis from the path which led to it. And so, my list of acknowledgements is a long one. Take solace, however, that this is a thesis and not an after-dinner speech, and so you have the opportunity to skip these pages entirely and hurry on to the “proper” content of this piece of writing! But, if you wish to see the names of those who have helped bring me to this point in my career and my life, read on...

I fear, in writing this, that I will have forgotten someone who ought to be included. If you’re searching for your name, and do not find it, please blame my somewhat-fickle memory and not any lack of appreciation for you and your support. It seems almost inevitable that these lists miss some important names which they ought to, by all rights, include. I’ve tried my best to be comprehensive, but I doubt I’ll escape the same fate as others.

But now, to start at the beginning...

Thanks to my mother, who taught me to listen, to read, and to learn. And to my father, who taught me to question, and to look out for the unobserved things in everyday

life. Thanks to both of them, for teaching me to always try my hardest without ever making me feel like I was under pressure. Whatever success I have had in this life is due to the beginnings which they gave me. Thanks also to my sisters, Lorraine, Sheila, and Deirdre, whose love, support, and advice have been unfailing. Especially to Sheila, who always encouraged my questions and imagination, and who taught me to pronounce “abominable!”

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I cannot imagine a better undergraduate experience than the one I had at Simon Fraser. The physics and math courses I took there left me well-prepared for the rigours of graduate school, both in their breadth and in their depth. Again, a few names stand out in my mind. Thanks to Brian Thompson for teaching me the wonderful mysteries and discoveries inherent in mathematics. Thanks to Tony Arrott, for inspiration and for being a wonderful example of how to be a physicist with style and panache. Thanks to Leigh Palmer, for a good start in statistical mechanics — that course will stand out in my memory as the best I’ve ever had. Thanks to Albert Curzon, for rigour and compassion. Thanks to Partha, for stealing his child’s “tippy top,” and for being a wonderful teacher. Thanks to John Cochran, another wonderful teacher. I will always cherish the memories of the time he spent in the lab with me. I don’t believe I have ever had a better learning experience. And thanks to Bill Plischke, for good advice whenever I most needed it.

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Before starting grad school, I took a year off. In retrospect, perhaps I should have spent that year pumping gas, or hauling coffee, or being a liftee. But instead, I had the wonderful experience of working for Barbara Frisken. Barb, thanks for all the good times, for all I learned from you about setting up and working in a physics lab, and for good advice over the years. Thanks, too, for what I can only assume must have been one hell of a letter of recommendation!

Thus far, I have acknowledged those who taught me in some sort of academic or research setting. But I have been molded and taught by many people of exceptional character whom I have been fortunate enough to call my friends. I have found out, over the years, that many people found high school to be a thoroughly unenjoyable experience. That could not have been further from the truth for me. "Thank you" to all of you who went through those years with me and made it such a great experience, especially those of you with whom I have been fortunate enough to keep in regular contact. In particular, to Sarah B., to Ed, to Ted, and to Kev.

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Though I have travelled far from Vancouver, those people always remain in my mind and in my heart.

Graduate school is not the easiest of times — especially those first two years. Although my emotional memories of that time have faded, and perhaps (I hope not!) I will eventually look back on them with rose-tinted glasses, I know that the only adjective that ever came close to describing those days was “hellish.” I’m sure that they would have been unbearable had I not been again blessed with the company of the incredible people who were stuck in the same boat as me. Thank you all. Aside from grad school (and sometimes, including it!), Boulder has been fun. Thanks to Orion, to Eric, to Keith, to Kristan, to Jim, to Julie, to Kim, and to all my other Boulder friends. Thanks to my roommates over the years — especially to Mas, Scott, and Rachel O. And to Julie, and Rachel P. — you helped me inestimably just by being yourselves at a time when I really needed that.

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I’ve never been a terribly single-minded person. In some ways, that has made grad school — especially in the world of JILA — very challenging. But I’ve always believed that even people with a broad range of interests may have something significant to contribute to physics. Time will tell. Nonetheless, I’d like to offer my thanks to those who created spaces for the “other” aspects of my self, and who shared in my expression of them. Thanks to everyone at UGGS, especially Tim, Adam, and Glenda. Thanks, as well, to those on the CU-UCSU JCSP (or DC, or CC, or SPC, or any of its other incarnations!), but especially to Susan Stafford, Kathryn Moerke (on whom I could always count for well-considered opinions), Ron Stump, and to Ali Vogt. Thanks to the wonderful people at the Mountain Sun, and to the folks at Trident. You people saved me...

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