

Appendix C

Mathematical Miscellany for SU(1,1) Berry's Phase

C.1 Calculation of SU(1,1) Berry's Phase Integral

We wish to calculate the integral

$$-\gamma = \int_{k_1=0}^{\sinh r} \int_{k_2=0}^{\sinh r} \frac{dk_1 dk_2}{\sqrt{1+k_1^2+k_2^2}}, \quad (\text{C.1})$$

which gives the geometric (anholonomic) contribution to the phase produced by a sequence of four squeezings constituting a closed loop on the unit hyperboloid (see Sec. 9.3). The first integral is straightforward, and so we immediately obtain:

$$-\gamma = \int_{k_2=0}^{\sinh r} \operatorname{arcsinh} \left[\frac{\sinh r}{\sqrt{1+k_2^2}} \right] dk_2. \quad (\text{C.2})$$

With the substitution $k_2 = \sinh x$, this becomes

$$-\gamma = \int_0^r \operatorname{arcsinh}[\sinh r \operatorname{sech} x] \cosh x dx. \quad (\text{C.3})$$

Identifying $dv = \cosh x dx$ and $u = \operatorname{arcsinh}[\sinh r \operatorname{sech} x]$ and integrating by parts, this becomes:

$$\begin{aligned} -\gamma &= \operatorname{arcsinh}[\tanh r] \sinh r - \int_0^r \frac{\sinh r \sinh^2 x}{\cosh x \sqrt{\sinh^2 r + \cosh^2 x}} dx \\ &= \operatorname{arcsinh}[\tanh r] \sinh r - \mathcal{I}. \end{aligned} \quad (\text{C.4})$$

In order to integrate \mathcal{I} , let $y = \cosh x$. In this case, the integral becomes:

$$\mathcal{I} = \sinh r \int_1^{\cosh r} \frac{1}{y} \sqrt{\frac{1-y^2}{\sinh^2 r + y^2}} dy. \quad (\text{C.5})$$

This integral is somewhat obscure. However, rewriting it as

$$\mathcal{I} = i \sinh r \int_1^{\cosh r} \frac{1}{y} \sqrt{\frac{1-y^2}{y^2 - (i \sinh r)^2}} dy \quad (\text{C.6})$$

puts it in a form found on pg. 362 of Ref. [228]. Thus,

$$\mathcal{I} = -\sinh r \operatorname{arctanh} \left[\frac{\sinh r}{\sqrt{\cosh^2 r + \sinh^2 r}} \right] - i \operatorname{arctanh} \left[\frac{i \sinh^2 r}{\sqrt{\cosh^2 r + \sinh^2 r}} \right]. \quad (\text{C.7})$$

Using the identities $\operatorname{arctan}(ix) = i \operatorname{arctanh}(x)$, $\operatorname{arctanh}(x) = \operatorname{arcsinh}\left(\frac{x}{\sqrt{1-x^2}}\right)$, and $i \operatorname{arcsinh}(x) = \operatorname{arcsin}(ix)$, this may finally be put in the form:

$$\mathcal{I} = -\operatorname{arcsinh}[\tanh r] \sinh r + \operatorname{arcsin}[\tanh^2 r], \quad (\text{C.8})$$

which gives

$$-\gamma = 2 \operatorname{arcsinh}[\tanh r] \sinh r + \operatorname{arcsin}[\tanh^2 r], \quad (\text{C.9})$$

which is Eq. (9.60).