Appendix A

Derivation of Group Velocity with Kerr Nonlinearity

To find the group velocity of an ultrashort pulse in the presence of the Kerr nonlinearity, we first express the electric field in terms of the slowly varying field amplitude $A(\boldsymbol{r}, t)$.

$$E(\boldsymbol{r}, t) = A(\boldsymbol{r}, t)\mathrm{e}^{\mathrm{i}(k_0 z - \omega_c t)} + \text{c.c.} \tag{A.1}$$

where ω_c is the center frequency of the pulse bandwidth and $k_0 = n_0\omega_c/c$ is the linear part of the wavevector at ω_c. $A(\boldsymbol{r}, t)$ satisfies the general form of the nonlinear Schrödinger equation given in Eqn. (13.2.25) in [8].

$$\left[\left(1 + \frac{\mathrm{i}}{\omega_c}\frac{\partial}{\partial \tau}\right)^{-1} \nabla_\perp^2 + 2\mathrm{i}k_0\frac{\partial}{\partial z'} + 2k_0 D\right] A(\boldsymbol{r}, t) = -\frac{4\pi\omega_c^2}{c^2}\left(1 + \frac{\mathrm{i}}{\omega_c}\frac{\partial}{\partial \tau}\right) p \tag{A.2}$$

In this equation, D is the differential operator

$$D = \sum_{n=2}^{\infty} \frac{1}{n} k_n \left(\mathrm{i}\frac{\partial}{\partial \tau}\right)^n \tag{A.3}$$

where k_n indicates the nth derivative of the linear part of the wavevector with respect to ω, evaluated at ω_c. D represents second- and higher-order dispersion. p is the slowly varying amplitude of the polarization of the material through which the pulse is propagating. Finally, Eqn. (A.2) is expressed in a retarded time frame specified by

$$z' = z \quad \text{and} \quad \tau = t - k_1 z \tag{A.4}$$

To calculate the group velocity we can ignore diffraction of the pulses, which is equivalent to dropping the $\nabla_\perp^2$ term in Eqn. (A.2). In a mode-locked laser we can also ignore higher-order dispersion and drop the D term. The Kerr nonlinearity in a material is produced by a third-order nonlinearity. Therefore, including the effects of dispersion up to first-order, we have for p

$$p = 3\left[\chi^{(3)}(\omega_c) + \frac{\mathrm{d}\chi^{(3)}}{\mathrm{d}\omega}\mathrm{i}\frac{\partial}{\partial\tau}\right]|A|^2 A \tag{A.5}$$

which is given in Eqn. (13.2.28) in [8]. Using this in Eqn. (A.2), we have

$$\begin{aligned} 2\mathrm{i}k_0\frac{\partial}{\partial z'}A(\boldsymbol{r},t) &= -\frac{12\pi\omega_c^2}{c^2}\left(1+\frac{\mathrm{i}}{\omega_c}\frac{\partial}{\partial\tau}\right)\left[\chi^{(3)}(\omega_c) + \frac{\mathrm{d}\chi^{(3)}}{\mathrm{d}\omega}\mathrm{i}\frac{\partial}{\partial\tau}\right]|A|^2 A \\ &= -\frac{12\pi\omega_c^2}{c^2}\left\{\chi^{(3)}(\omega_c) + \mathrm{i}\left[\frac{\mathrm{d}\chi^{(3)}}{\mathrm{d}\omega} + \frac{1}{\omega_c}\chi^{(3)}(\omega_c)\right]\frac{\partial}{\partial\tau}\right\}|A|^2 A \end{aligned} \tag{A.6}$$

where the term higher than first-order in $\partial/\partial\tau$ has been dropped. From this we can write

$$\frac{\partial}{\partial z'}A(\boldsymbol{r},t) = \frac{6\pi\omega_c^2}{k_0c^2}\left\{\mathrm{i}\chi^{(3)}(\omega_c) - \left[\frac{\mathrm{d}\chi^{(3)}}{\mathrm{d}\omega} + \frac{1}{\omega_c}\chi^{(3)}(\omega_c)\right]\frac{\partial}{\partial\tau}\right\}|A|^2 A \tag{A.7}$$

Now, to obtain the propagation equation for the intensity envelope of the pulse from Eqn. (A.7), consider first that

$$\frac{\partial}{\partial\tau}|A|^2 A = 2AA^*\frac{\partial A}{\partial\tau} + A^2\frac{\partial A^*}{\partial\tau} \tag{A.8}$$

and so

$$\begin{aligned} A^*\frac{\partial}{\partial\tau}|A|^2 A &= 2|A|^2 A^*\frac{\partial A}{\partial\tau} + |A|^2 A\frac{\partial A^*}{\partial\tau} \\ &= |A|^2 A^*\frac{\partial A}{\partial\tau} + |A|^2\frac{\partial}{\partial\tau}|A|^2 \end{aligned} \tag{A.9}$$

Multiplying Eqn. (A.7) by A^* and using Eqn. (A.9) produces

$$\begin{aligned} A^*\frac{\partial}{\partial z'}A(\boldsymbol{r},t) = \frac{6\pi\omega_c^2}{k_0c^2}\mathrm{i}\chi^{(3)}(\omega_c)|A|^4 \\ -\frac{6\pi\omega_c^2}{k_0c^2}\left[\frac{\mathrm{d}\chi^{(3)}}{\mathrm{d}\omega} + \frac{1}{\omega_c}\chi^{(3)}(\omega_c)\right]\left[|A|^2 A^*\frac{\partial A}{\partial\tau} + |A|^2\frac{\partial}{\partial\tau}|A|^2\right] \end{aligned} \tag{A.10}$$

Adding each side of this equation to its complex conjugate provides the propagation equation for the intensity envelope.

$$\frac{\partial}{\partial z'}|A|^2 = -\frac{18\pi\omega_c^2}{k_0c^2}\left[\frac{\mathrm{d}\chi^{(3)}}{\mathrm{d}\omega} + \frac{1}{\omega_c}\chi^{(3)}(\omega_c)\right]|A|^2\frac{\partial}{\partial\tau}|A|^2 \tag{A.11}$$

To determine the group velocity, which corresponds to the velocity of propagation of the intensity envelope, we need to express Eqn. (A.11) in terms of coordinates in the laboratory frame instead of the retarded time frame. From Eqn. (A.4), we have

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z} + k_1 \frac{\partial}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} \tag{A.12}$$

With this coordinate transformation, the propagation equation in the laboratory frame becomes

$$\frac{\partial}{\partial z}|A|^2 + \left\{ k_1 + \frac{18\pi\omega_c^2}{k_0 c^2}\left[\frac{\mathrm{d}\chi^{(3)}}{\mathrm{d}\omega} + \frac{1}{\omega_c}\chi^{(3)}(\omega_c)\right]|A|^2 \right\} \frac{\partial}{\partial t}|A|^2 = 0 \tag{A.13}$$

This differential equation has the form $\frac{\partial}{\partial z}|A|^2 + \gamma\frac{\partial}{\partial t}|A|^2 = 0$, which has a solution of the form $|A|^2 \sim \mathrm{e}^{\mathrm{i}(z-\gamma^{-1}t)}$ that propagates at the velocity γ^{-1}. Therefore, from Eqn. (A.13) the group velocity, v_g, can be identified as

$$v_g = \left\{ k_1 + \frac{18\pi\omega_c^2}{k_0 c^2}\left[\frac{\mathrm{d}\chi^{(3)}}{\mathrm{d}\omega} + \frac{1}{\omega_c}\chi^{(3)}(\omega_c)\right]|A|^2 \right\}^{-1} \tag{A.14}$$

From Eqn. (4.1.16) in [8], the intensity of the pulse, I, is given by

$$I = \frac{n_0 c}{2\pi}|A|^2 \tag{A.15}$$

Also, the nonlinear coefficient of the index of refraction, n_2, is given by Eqn. (4.1.19) in [8].

$$n_2 = \frac{12\pi^2}{n_0^2 c}\chi^{(3)} \tag{A.16}$$

Using

$$k_1 = \left[\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{\omega n_0}{c}\right)\right]_{\omega_c} = \frac{n_0 + \omega_c\left(\frac{\mathrm{d}n_0}{\mathrm{d}\omega}\right)_{\omega_c}}{c} \tag{A.17}$$

the expression for the group velocity can be rewritten as

$$v_g = \frac{c}{n_0 + \omega_c\left(\frac{\mathrm{d}n_0}{\mathrm{d}\omega}\right)_{\omega_c} + 3\left[n_2 + \omega_c\left(\frac{\mathrm{d}n_2}{\mathrm{d}\omega}\right)_{\omega_c}\right] I} \tag{A.18}$$

where the term in the denominator proportional to $\omega_c(n_2/n_0)(\mathrm{d}n_0/\mathrm{d}\omega)_{\omega_c}I$ has been dropped since it is negligibly small for the parameters given in Section 3.1.