Appendix C

Derivation of the Dispersion Relation for the Relativistic Rayleigh-Taylor Instability

The derivation of the Rayleigh Taylor instability for a relativistic plasma by Allen and Hughes (1984) is based on the formalism used in Chandrasekhar (1961), using Eulerian rather than Lagrangian first order variables, which is not a particularly transparent derivation and is formally not self-consistent (thought is *does* produce the correct results). Since the dispersion relation given by Allen and Hughes (1984) contains typographical errors we found it appropriate to derive the fully relativistic dispersion relation for any ideal gas (i.e., for arbitrary adiabatic indices $\gamma_{\rm ad}$ and arbitrary ratios of random internal energy to rest mass energy density). It should also be noted that the incompressible analysis commonly used is only valid for wavelengths smaller compared to the scale height in the fluid, even in the non-relativistic case. The analysis by Mathews and Blumenthal (1977) assumes isothermal perturbations, which is not a realistic assumption in the cases we are interested in.

The equation of motion is given by $\Gamma T^{i\beta}{}_{,\beta} - u^i T^{0\beta}{}_{,\beta} = 0$, where we assumed the Einstein summation convention, i runs from 1 to 3. u^α is the four-velocity, T the stress-energy tensor, and Γ the fluid Lorentz factor. After a little algebra, this can be written as

$$\Gamma^2 w \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right] \left(\mathbf{v}/c^2 \frac{\partial p}{\partial t} + \nabla p\right) = 0. \tag{C.1}$$

Here, $w = e + p$ is the enthalpy, e is the internal energy, p is the pressure, and ρ the rest mass density, all measured in the fluid rest frame. For polytropic gases, $e = 1/(\gamma_{\rm ad} - 1)p$ where $\gamma_{\rm ad}$ is the ratio of specific heats; $\mathbf{v}$ is the three-velocity, $v^{\rm i} = u^{\rm i}/\Gamma$, i from 1 to 3.

The continuity equation for a conserved particle number density n can be written as

$$\Gamma\left(\frac{\partial}{\partial t}+\nabla\right)n+n\left(\frac{\partial\Gamma}{\partial t}+\nabla(\Gamma\mathbf{v})\right)=0, \tag{C.2}$$

while the equation of state is

$$\Gamma\left(\frac{\partial}{\partial t}+\nabla\right)p+\gamma p\left(\frac{\partial\Gamma}{\partial t}+\nabla(\Gamma\mathbf{v})\right)=0. \tag{C.3}$$

All of these equations are fully general within the limits of special relativity (i.e., weak gravity).

Specializing to Rayleigh-Taylor instability, we assume that a gravitational field is present equivalent to the acceleration of the plasma in the GRB. We will assume that a planar contact discontinuity exists in the fluid, placed at $z = 0$. Gravity is assumed to act perpendicularly to that surface. The background flow is compressible, so for self-consistency we must assume that the density and pressure of the background flow are stratified. For mathematical simplicity we will assume that the gas has an exponential dependence and is isothermal on either side of the discontinuity. Without loss of generality, we can also specify that the x-axis lie along the direction of the wave vector $\mathbf{k}$.

Assuming small perturbations in the dynamical quantities, we can linearize the equations. We can then Fourier transform the linearized equations in x and t and transform to a coordinate system in which the perturbed interface is at rest. The new coordinates are (Mathews and Blumenthal 1977):

$$\chi\equiv x-i\frac{n}{k}t$$

$$\zeta\equiv z-\xi=z-\xi_0\exp{(ikx+nt)}, \tag{C.4}$$

where ξ is the (first order) displacement of the interface.

The perturbed equations then take the form

$$n\tilde{\rho}+\rho(ikv+\frac{\partial w}{\partial\zeta})+(w-n\xi)\frac{\partial\rho}{\partial\zeta}=0$$

$$n\tilde{p}+\gamma_{\mathrm{ad}}p(ikv+\frac{\partial w}{\partial\zeta})+(w-n\xi)\frac{\partial p}{\partial\zeta}=0$$

$$nhv+ik\tilde{p}-ik\xi\frac{\partial p}{\partial\zeta}=0$$

$$nhw+\frac{\partial\tilde{p}}{\partial\zeta}+g\tilde{h}=0 \tag{C.5}$$

Here, a ˜ indicates a first order quantity, v and w are the (first order) velocity in the x and z directions, and $n \equiv i\omega$, where ω is the temporal frequency.

The equations also indicate that the ζ dependence of all first order quantities is the same. Furthermore, the zeroth order quantities also have the same ζ dependence (by assumption). A little algebra then gives

$$\begin{aligned}\frac{\partial^2 w}{\partial \zeta^2} &- \left[g\frac{h}{p} - \frac{1}{c^2}\right]\frac{\partial w}{\partial \zeta} \\ &- \left[\left(1 - \frac{1}{\gamma_{\mathrm{ad}}}\right)\frac{k^2 g^2}{n^2}\frac{\rho}{p} + \left(\frac{hn^2}{p\gamma_{\mathrm{ad}}} + k^2\right)\right] w = 0\end{aligned} \tag{C.6}$$

This equation has the well known solution

$$w = C_+ \exp K_+\zeta + C_- \exp K_-\zeta \tag{C.7}$$

where C_+ and C_- are constants to be determined later and

$$\begin{aligned}K_\pm &= \frac{hg}{2p} - \frac{g}{2c^2} \\ &\pm\sqrt{\left[\frac{hg}{2p} - \frac{g}{2c^2}\right]^2 + \left(1 - \frac{1}{\gamma_{\mathrm{ad}}}\right)\frac{k^2 g^2}{n^2}\frac{\rho}{p} + \left(\frac{hn^2}{p\gamma_{\mathrm{ad}}} + k^2\right)}\end{aligned} \tag{C.8}$$

Since the energy contained in the perturbation must be finite, we must chose the constants $C_\pm$ such that the solution decays exponentially away from the interface. In other words, the real part of K must be negative (positive) for positive (negative) ζ.

The equations of motion dictate that the pressure be continuous across the discontinuity. Equating the first order pressure on both sides finally leads to the dispersion relation. A little algebra reveals

$$\tilde{p} = \frac{\left(\frac{k^2}{h}\xi\gamma_{\mathrm{ad}}p + n^2\xi - nw\right)\frac{\partial p}{\partial \zeta} - \gamma_{\mathrm{ad}} np\frac{\partial w}{\partial \zeta}}{n^2 + \frac{\gamma_{\mathrm{ad}} pk^2}{h}}. \tag{C.9}$$

The dynamic boundary condition is simply

$$\xi = \frac{1}{n}\left(w - vk\xi\right) = \frac{w}{n}, \tag{C.10}$$

dropping the second order term. This is derived from requiring that the fluid on both sides of the interface move parallel to each other, in other words, no fluid crosses the interface. Together, these equations yield

$$\frac{k^2 g h_1 + n^2 K_1 h_1}{\frac{n^2 h_1}{p_1} + k^2 \gamma_{\mathrm{ad},1}} = \frac{k^2 g h_2 + n^2 K_2 h_2}{\frac{n^2 h_2}{p_2} + k^2 \gamma_{\mathrm{ad},2}}. \tag{C.11}$$

Here the subscripts 1 and 2 denominate the two sides of the shock.