

Chapter 1

Introduction

This thesis is concerned with a class of nonintegrable systems from atomic physics: atoms in static, external electromagnetic fields. These systems present a challenge not found in their integrable¹ or near-integrable counterparts, such as the hydrogen atom or the low lying states of atoms and molecules. To illustrate the difficulty with nonintegrable systems I wish to imagine a dialogue between a diligent graduate student and her advisor. The graduate student has been hard at work in the lab taking photoabsorption spectra of an atom having a nonintegrable Hamiltonian. After months of building electronics and tweaking lasers, the graduate student has scans of a region of the spectrum with an excellent signal to noise ratio. The advisor enters the lab to see the new results.

“Yes, I received your email and I wanted to see some of the spectra you have taken.”

“Of course,” replies the graduate student, as she pulls out the lab notebook containing the newly obtained scans. “Here is the region from 109,700 to 109,900 wavenumbers. At first I thought the spectrum was all noise, but I have repeated the experiment over the same range three times and all of the features are reproducible.”

Slightly skeptical, the advisor puts on his eyeglasses to take a closer look at the different scans.

“Wow, they do look identical. I guess after all of the work you have done these might be real absorption peaks.”

Pleased by her advisor’s confidence, the student replies, “I think they are.”

It now becomes clear that the advisor is thinking about the physics of the spectrum. “You have

¹ A quantum system with N degrees of freedom is integrable if there exist N independent operators \hat{A}_i that commute with the Hamiltonian and with each other. This set of operators is sometimes referred to as a “complete set of commuting observables.”

been doing a literature search on this system, right? What do we know about this large absorption peak?"

"Which one?" asks the student.

"This one right here that stands out so strongly. You would think this peak would show up in the lower resolution experiments that have been performed previously."

"Oh, yes, I did find this article that mentions a large peak at that energy . . . at about 109793.5 wavenumbers."

Encouraged, the advisor inquires further, "well then what do we know about it?"

With a puzzled look the student hesitates "well . . . uhh . . . it is at that energy there, and it's that high."

"Well obviously, I can see that, but what else do we know?"

The student knows that she should say something, so she stretches, "um . . . it is right next to those two peaks?"

Just as the student is beginning to doubt the reliability of both her literature search and the experimental data, the advisor gets a light in his eye and proclaims,

"Oh, of course, you are right, that's all there is to know about that peak. This system is nonintegrable!"

The point of this dialogue is to emphasize that the only good quantum number of a strongly nonintegrable, autonomous system is its energy. This is in contrast to an integrable or partially integrable system, which has one or more good quantum numbers other than the energy. It is well known [1] that every good quantum number corresponds to a symmetry of the Hamiltonian. Each time a symmetry of a system is strongly broken, the quantum number associated with the symmetry is no longer useful for describing the eigenstates of the system. What is often not appreciated is that the quantum numbers of an eigenstate give us an intuitive picture of the physics of the state. Given the quantum numbers, we immediately have access to information about the nodal structure of the wavefunction and other issues such as degeneracies. Thus, quantum numbers are one of the main ways that we "see" quantum mechanical states. The main difficulty in nonintegrable systems is then in our ability to gain an intuitive picture of the quantum mechanics.

Perhaps the biggest advance towards understanding the quantum mechanics of multi-dimensional, nonintegrable systems has been the development of semiclassical approximations for the solutions of the Schrödinger equation. This is demonstrated in the following brief history of the study of atoms in strong magnetic fields, or diamagnetic atoms.

1.1 Historical background

At the beginning of the 20th century, the newly discovered quantum mechanics diverged from classical mechanics. As Einstein [2] and others realized, quantization using classical trajectories quickly ran into difficulties in multidimensional, nonintegrable systems. The invariant tori used to quantize integrable systems begin to break down as integrability is lost. While the WKB [3] quantization procedure for one dimensional systems and the multidimensional extension for integrable systems (EBK) [2, 4, 5, 6] had limited success, no such semiclassical quantization procedure could be found for nonintegrable systems.

To some extent, the status of nonintegrable systems in quantum mechanics at this point in history is not surprising. A similar impasse existed for nonintegrable classical systems. The difficulties for the classical case were elucidated by the work of Henry Poincaré [7]. The issue at the time was the prediction of the long time behavior of the solar system. Poincaré showed, to the dismay of many, that all classical perturbative expansions of the motion of the solar system contain irremovable singularities due to resonances. Without the benefits of modern computational power, Poincaré's theorem shattered the only available method of solution. At the level of both quantum mechanics and classical mechanics, progress on the understanding of nonintegrable systems slowed drastically for about fifty years.

In the 1950s and 1960s, work by Kolmogorov [8], Arnol'd [9, 10] and Moser [11] began to illuminate the nature of classical nonintegrability. The results of their work, known as the KAM theorem [12], gives a detailed account of exactly how the invariant tori in phase space break up as symmetries are broken. Also, beginning with the work of Edward Lorenz [13], computers began to give dramatic new insights into the nature of strongly nonintegrable classical systems. Essentially, classical chaos has been discovered. The breakup of invariant tori into finer and finer phase space structures could now be studied

in detail. It was seen that the invariant structures in the phase space of chaotic systems - periodic orbits - occupied infinitesimally small volumes of phase space. This seemed incompatible with one of the main ideas of quantum mechanics: that phase space volumes are limited by the fundamental constant \hbar through the Heisenberg uncertainty principle. However, as classical chaos was studied more thoroughly in the 1960s and 1970s, it was realized that semiclassical quantization of classically chaotic systems might be possible after all. The major breakthrough came with the work of Balian, Bloch and Gutzwiller. Balian and Bloch [14, 15, 16, 17] showed that oscillations in the density of states of electromagnetic cavities and quantum billiards could be understood in terms of classical periodic orbits. Gutzwiller [18, 19, 20, 21] elaborated on this idea through his derivation of a “trace formula” for the quantum density of states of a smooth nonintegrable Hamiltonian. With Gutzwiller’s derivation, classical mechanics reentered the realm of quantum mechanics for good.

Gutzwiller’s semiclassical trace formula for the quantum density of states introduced a new way of looking at the spectrum of Hamiltonians having chaotic classical dynamics. Excellent discussions of the trace formula can be found in [22, 23, 24]. In his approach, the density of states is broken up into a smooth, average part $\tilde{g}(E)$, and an oscillating part $\delta g(E)$:

$$g(E) = \sum_n \delta(E - E_n) = \tilde{g}(E) + \delta g(E). \quad (1.1)$$

The famous trace formula,

$$\delta g(E) = \frac{1}{\hbar\pi} \sum_{po} \frac{T_{po}}{\sqrt{\det(\tilde{M}_{po} - 1)}} \cos\left(\frac{1}{\hbar} S_{po}(E) - \sigma_{po} \frac{\pi}{2}\right), \quad (1.2)$$

gives a relationship between the oscillating part of the density of states $\delta g(E)$ and the classical periodic orbits of the system in the limit $S_{po} \gg \hbar$. These **periodic orbits** are solutions of the classical equations of motion that return to an initial point in phase space after a period T_{po} . On the right side of Eq. (1.2), the properties (action S_{po} , Maslov index σ_{po} , and stability matrix \tilde{M}_{po}) of these purely classical orbits are seen to provide all of the information about the quantum mechanical density of states. The only signature of the quantum world on the right side of Eq. (1.2) is the appearance of the constant \hbar . Although a similar trace formula for integrable systems has been derived by Berry [25, 26], the result of Gutzwiller applies to chaotic systems where the periodic orbits are well isolated in phase space.

The physics in the trace formula (1.2) is manifested when the delta functions $\delta(E - E_n)$ in the exact density of states (1.1) are smoothed over using some convolution function of width ΔE . The resulting smoothed density of states shows dramatic oscillations with energy. The great advance of the trace formula (1.2) is to allow the interpretation of these oscillations in terms of the classical periodic orbits having $T_{po} < \frac{2\pi\hbar}{\Delta E}$. Thus, entire sequences of energy smoothed eigenstates of nonintegrable Hamiltonians can be interpreted with only a few classical periodic orbits of the corresponding classical system. This represents a huge improvement over interpreting the density of states by saying “this eigenstate has an energy of . . . and it is next to this one, this one and this one.”

Atoms in external magnetic fields represent one of the most important examples of this type of analysis. The experiments of Garton and Tomkins [27, 28, 29] were the first to show interesting new physics in the spectra of diamagnetic atoms. Their major discovery was that the near threshold photoabsorption spectra of atoms in strong (0-6 Tesla) magnetic fields show dramatic oscillations with energy, which are independent of the atom being studied. The large spacing of these “quasi-Landau” resonances $\frac{3}{2}\hbar B$ (B is the magnetic fields in a.u.), as Edmonds [30] and Starace [31] elucidated, is related to a classical orbit of the Rydberg electron having period $\frac{2}{3} \left(\frac{2\pi}{\hbar B} \right)$. This classical orbit, the quasi-Landau orbit, begins at the nucleus, travels out perpendicularly to the magnetic field and returns to the nucleus after deflecting off the magnetic field. It is ironic that the observation of these global oscillations depended critically on the poor resolution of the experimental spectrum; high resolution spectra recorded later (see [32] for example), when experimental methods improved, show dense sequences of seemingly random absorption lines.

Soon thereafter, experiment and theory showed that the quasi-Landau oscillations were merely the tip of the iceberg. Higher resolution experiments on hydrogen in a 5.96 Tesla field [33, 34, 35] revealed the contributions of additional, longer period classical orbits of the highly excited electron. This experimental work by Welge’s group in Bielefeld, Germany demonstrated that the contribution of each such orbit to the photoabsorption cross section could be extracted by taking the Fourier transform of the experimental spectrum. The resulting **recurrence spectrum** shows strong peaks in the time domain at the periods of these newly uncovered classical orbits. A quantitative theory of the recurrence spectrum

was first provided by Du and Delos² [37, 38]. This theory and its extensions are known as **closed-orbit theory**.

Closed-orbit theory echoes many of the ideas of Gutzwiller's trace formula (1.2). Like the trace formula for the density of states, the photoabsorption cross section $\sigma(E)$ in closed-orbit theory [38] is written in terms of an average part $\tilde{\sigma}(E)$ and an oscillating part $\delta\sigma(E)$:

$$\sigma(E) = \tilde{\sigma}(E) + \delta\sigma(E). \quad (1.3)$$

Using semiclassical wavefunctions away from the nucleus, Du and Delos showed that the oscillating part of the photoabsorption cross section³ can be written as a sum over the closed classical orbits of the atomic electron in an external field:

$$\delta\sigma(E) = 8\pi^2\alpha\omega \sum_{co} A_{co} \cos\left(S_{co} - \sigma_{co}\frac{\pi}{2} + \frac{3\pi}{4}\right). \quad (a.u.) \quad (1.4)$$

As in the trace formula (1.2), the phase of each oscillating term is determined by the classical action S_{co} and Maslov index σ_{co} of each closed orbit. The amplitude A_{co} involves both properties of the classical orbit of the electron (its classical stability and initial and final polar angles) along with properties of the initial quantum state of the atom (dipole matrix elements). The **closed orbits** that determine the physics of $\delta\sigma(E)$ in a semiclassical approximation are classical trajectories of the Rydberg electron that are launched radially outward from the nucleus, scatter off the long range Coulomb and magnetic field and then return radially to the nucleus. Closed orbits, rather than periodic orbits, are relevant in photoabsorption experiments because the initial atomic state is strongly localized near the nucleus.

For light atoms in external magnetic and electric fields, closed-orbit theory has proven to be a quantitative and elegant method of calculating and interpreting recurrence spectra. Over the past two decades, multiple generations of experiments have measured the recurrence spectra of hydrogen [39], lithium [40] and helium [41, 42, 43, 44, 45] in strong magnetic fields. Almost universally, the agreement of these experiments with the predictions of closed-orbit theory has been spectacular; both the location and amplitude of recurrence peaks are predicted to within a few percent. In addition, the closed orbits

² A similar treatment was developed simultaneously by Bogomolny [36] although his approach has not received as much attention.

³ Practitioners of closed-orbit theory often use an oscillator-strength density $Df(E)$ instead of the atomic absorption cross section. The two are related by the formula $\sigma(E) = 2\pi^2\alpha Df(E)$.

underlying each recurrence peak provide a simple interpretation of the time domain physics. Similar agreement is shown in the Stark recurrence spectra of these light atoms subjected to a static electric field [46, 47, 48, 49]. To achieve this level of agreement with experiment, two extensions of closed-orbit theory have been necessary.

First, the effects of a nonhydrogenic ionic core have been included. Following quantum-defect theory [50], the electron-core interactions are characterized by a set of energy independent quantum defects. When combined with semiclassical wavefunctions away from the core [51, 38], these quantum defects permit an extension of closed-orbit theory to single-channel atoms. Such results, obtained by Dando *et al.* [52, 53] and by Shaw and Robicheaux [52, 53, 54], predict the emergence of new recurrence peaks, called core-scattered recurrences, when the quantum defects are turned on (see also [55]). These appear as a result of one primitive closed orbit of period T_1 scattering into another of period T_2 to produce a new peak at the combined period $T_1 + T_2$. For helium [56, 45] and lithium [40], experiments have confirmed the existence of these nonclassical core-scattered features.

Second, artificial divergences associated with bifurcations of closed orbits have been regularized to give a uniform semiclassical approximation [57, 58, 59]. As the external field strength or the energy of the Rydberg electron is increased, bifurcations of the closed orbits occur [39]. These bifurcations cause well known divergences in the semiclassical amplitude A_{co} in Eq. (1.4) at the points where new orbits come into existence. This effect is unphysical as the exact quantum recurrence spectrum is finite everywhere. Delos and coworkers [60, 61] have used normal form theory to investigate the basic types of bifurcations present in diamagnetic atoms. Because each type of bifurcation has a different topology in phase space, a general, uniform semiclassical theory has proven difficult. In spite of this, some progress has been made. Gao and Delos [62] have given a uniform approximation for the bifurcations of certain classes of orbits in an external electric field; those parallel to the field, the “uphill” and “downhill” orbits. Using these results, Shaw and Robicheaux [54] have given the most promising generalization of closed-orbit theory to date, which incorporates both bifurcations and core-scattering for Stark recurrence spectra. The validity of their formulation has been verified by accurate quantum calculations and a recent experiment [49]. For the case of atoms in magnetic fields, the only work on a uniform semiclassical

treatment has been by Main and Wunner [63]. While suggestive, their approach contains additional unphysical divergences below the bifurcation points that must be dealt with (i.e. canceled by hand or ignored). Additionally, their theory has not yet been tested critically. Thus, while there have been some spectacular successes in regularizing bifurcations in closed-orbit theory, much work remains to be performed in this area. For the most part, however, the inclusion of core-scattering and bifurcations into closed-orbit theory enables the prediction of recurrence spectra of light atoms.

Heavier atoms, however, have proven difficult for closed-orbit theory. Thus far, the success of closed-orbit theory has been limited to atoms with at most two nonzero quantum defects. Recent experiments on barium [64, 65, 66] and argon [67] in electric fields show dramatic differences from the predictions of closed-orbit theory. Even when the core-scattering effects described above are included for these atoms, agreement remains dismal. Furthermore, it appears that the presence of three nonzero quantum defects (as in rubidium) causes the expansions of Dando *et al.* [53] and Shaw and Robicheaux [54], which work beautifully for helium and lithium, to diverge. Thus, the presence of multichannel ionic cores and multiple ionization thresholds seem to present a fundamental difficulty for semiclassical approaches.

The difficulty then is the short range interaction between the Rydberg electron and a multichannel positive ion. This is somewhat ironic given the success of multichannel quantum-defect theory [68] in treating this physics. Since its introduction by Seaton [50] in the 1950s, multichannel quantum-defect theory (MQDT) has become one of the mainstays of modern atomic theory. In MQDT, the quantum defects are generalized into a short-range scattering matrix $\underline{S}^{\text{core}}$, which fully characterizes the scattering of the Rydberg electron from the ionic core. The multiple ionization thresholds and inelastic electron-ion scattering characteristic of complex atoms are all handled accurately and elegantly in this fully quantum-mechanical approach. However, because MQDT requires a simple long range potential, a new approach must be found when external fields destroy the simplicity of the electron's motion far from the nucleus.

Thus, there exists a dilemma in the theory of atoms in static, external magnetic and electric fields. While semiclassical methods, such as closed-orbit theory, provide an efficient and elegant way of treating the motion of a Rydberg electron far from the nucleus, they fail when the electron is within a few

Bohr radii of the ionic core of a multichannel atom. On the other hand, quantum-defect theory handles this short range physics without difficulty - but only when the long range physics is integrable. An understanding of multichannel atoms in nonintegrable configurations of external electric and magnetic fields requires that both the short range and long range physics of the Rydberg electron are treated accurately.

One way out of the difficulties (core-scattering, bifurcations) involved in semiclassical approximations is to solve the Schrödinger equation exactly. This approach has been taken by a number of researchers [69, 70, 71, 72, 73, 74] and is important to mention. These methods, which involve large scale quantum-mechanical calculations, have progressed through a combination of increased computer power and efficient algorithms for solving the Schrödinger equation. Typically, a variational approach such as *R*-matrix theory, along with an expansion of the wavefunction in a basis set (B-splines, Sturmians, finite elements), is used to convert the multidimensional Schrödinger equation to a matrix diagonalization or else to the solution of an inhomogeneous linear system of equations. Accurate recurrence spectra have been calculated for atoms in strong magnetic fields (1-10000 Tesla) using these techniques and show excellent agreement with experiments [32, 43, 44]. Successful applications to date include a number of single channel atoms in magnetic fields, such as the alkali-metal atoms [71, 74, 72, 75], and Ba and Sr [71, 76, 77] at their lowest thresholds. Similar calculations have been performed for multichannel atoms molecules in electric fields [78, 79, 80, 81, 82] using the methods of Harmin [83] and Fano [84]. Here, the long range physics is simpler than the magnetic field case because motion of the Rydberg electron in the combined Coulomb and electric fields is separable in parabolic coordinates.

While these fully quantum mechanical approaches accurately predict the recurrence spectra of many atoms in external magnetic and electric fields, their usefulness remains limited. Unlike closed-orbit theory, exact quantum calculations struggle to yield physical insight into the spectra they provide. We can **predict** the spectra, but the development of qualitative understanding is difficult or seemingly impossible using fully quantum approaches. For integrable systems, this difficulty is overcome by labeling the quantum states with quantum numbers. However, as I have emphasized in this Introduction, quantum number other than “energy” are useless in strongly nonintegrable systems such as atoms in magnetic

fields. Thus, as experiments begin to probe multichannel atoms in external fields, interpretation of the photoabsorption spectra remains the most difficult issue. Examples of this difficulty are provided by recent experiments on Ba [65, 66] and Ar [67] in electric fields, where simple features in the recurrence spectra remain uninterpreted to a large degree.

1.2 Outline of the results

In this thesis I develop a unified theory of complex atoms in external electric and magnetic fields. Using the ideas and methods from both multichannel quantum-defect theory and closed-orbit theory, I describe a complete picture of the photoabsorption process. In such a process, the atomic electron is moved from an initial state $|\psi_0\rangle$ at energy E_0 to a final state $|\psi_f\rangle$ having energy $E = E_0 + \hbar\omega$, after absorbing a photon of frequency ω . Determining the final state wavefunction $|\psi_f\rangle$, is the main task in any calculation of the photoabsorption cross section. In this thesis, I obtain this final state in a roundabout manner. As quantum-defect theory shows, all of the information contained in the quantum state $|\psi_f\rangle$ can be repackaged into one or more scattering matrices. This reformulation of the electron's final state leads to a simple physical picture of the electron's motion. Because every derivation and formula contained in this thesis relies on this physical picture, it is useful to present the picture here:

The state reached by an atomic electron in a photoabsorption experiment is a combination of two time-independent scattering processes. In the first, the electron is launched outward from the nucleus, scatters off the long range fields, and then returns to the nucleus. In the second, the electron travels inward to scatter off the residual ionic core.

Chapters 2 and 3 present this physical picture and a mathematical description that applies to any atom in any configuration of static external electric and magnetic fields. After reviewing the important elements of quantum-defect theory, I introduce two scattering (or S) matrices: one for the scattering of the electron off the long range fields $\underline{S}^{\text{LR}}$, and another for the short range electron-core scattering $\underline{S}^{\text{core}}$. These two scattering matrices completely determine the Rydberg electron's bound state energy eigenvalues. I derive both a quantization condition and a normalization condition for the bound states in terms of the S -matrices. The result is a completely general method for quantizing multichannel atoms

in external fields. Connections with previous treatments such as Harmin's theory of the Stark effect, and Bogomolny's semiclassical quantization scheme are elucidated.

In Ch. 3 a relationship between the S -matrices and the total photoabsorption cross section is derived. Rather than focusing on the infinite resolution spectra, I smooth over the details of individual absorption lines and explore the energy-smoothed spectrum instead. This approach is inspired by the results of closed-orbit theory, and shows that an exact quantum mechanical generalization of closed-orbit theory can be derived in terms of S -matrices. In addition, the result also connects with familiar formulas of quantum-defect theory. The final formula for the photoabsorption cross section, while still an exact quantum-mechanical result, contains much of the physical insight of semiclassical methods like closed-orbit theory. Again, I emphasize that the results of Chs. 2 and 3 are completely general, applying to any atom in any configuration of external electromagnetic fields. Beginning with Ch. 4, however, I specialize to the case of single channel atoms in external magnetic fields. While multichannel atoms can, in principle, be included into my formulation of the photoabsorption process, a number of subtle features about single channel atoms must be understood first.

Chapters 4 and 5 give the details of how the long-range scattering matrix $\underline{S}^{\text{LR}}$ can be calculated in either a fully quantum mechanical or semiclassical framework. First, In Ch. 4, the methods of variational R -matrix theory are extended to calculate an accurate quantum $\underline{S}^{\text{LR}}$. While these calculations are based on familiar techniques in atomic theory, a few extensions are needed. The most significant of these is the analytic continuation of the long-range S -matrix to complex energies. This is needed to produce the energy smoothed cross section of Ch. 3 and can be accomplished within the framework of R -matrix theory without difficulty. I end Ch. 4 by presenting calculations for an atom in a static magnetic field that implement the methods of the chapter. These calculations show that the long-range S -matrix can be analyzed in the spirit of closed-orbit theory by taking the Fourier transform, or recurrence strength, of its matrix elements. This allows the detection of nonclassical paths, or ghost orbits, of the electron as it scatters off the long-range fields.

In Ch. 5, I introduce semiclassical approximations into my S -matrix theory of photoabsorption. More specifically, I use a semiclassical Green's function to derive a semiclassical long-range S -matrix

for the motion of an atomic electron in a static magnetic field. By writing $\underline{S}^{\text{LR}}$ as matrix elements of an energy domain Green's function, a versatile approach for deriving semiclassical approximations to the S -matrix is achieved. This treatment allows a detailed study of how the closed-orbits are selected to contribute to the recurrence spectra. The generality of my method is demonstrated as it is used to treat a number of special cases where the primitive semiclassical approximation fails, such as near the bifurcations of closed orbits. My results both reproduce and extend the usual treatment of closed-orbit theory.

Chapter 6 uses the semiclassical S -matrices of Ch. 5 and the preconvolved photoabsorption cross section of Ch. 3 to develop a semiclassical theory for the photoabsorption rate. After the failure of a naive approach to the semiclassical approximation is outlined, I use accurate quantum S -matrices to uncover an important relationship between core-scattered orbits, and other nonclassical orbits called ghost orbits. When this relationship is put into mathematical terms, an improved, resummed semiclassical theory can be developed. In contrast to previous semiclassical theories for nonhydrogenic atoms, my result is generally convergent when more than one quantum defect is large. After deriving the final result, it is applied to a number of test cases, including lithium and rubidium in an external magnetic field.

Atomic units ($e = m = \hbar = 1$, $c = 137$) will be used throughout this dissertation unless otherwise stated. One atomic unit of magnetic field is equal to 2.35×10^5 Tesla.

1.3 Scaled variable recurrence spectroscopy

One of the most significant techniques in the study of atoms in external electric and magnetic fields is the use of so-called scaled variables. Because I will use these scaled variables throughout this thesis, their main features are summarized here. A detailed description for the case of hydrogen in static magnetic field can be found in Appendix A.

In standard spectroscopy, cross sections are measured or calculated as a function of the energy E . When an external electromagnetic field is applied to the system under consideration, the field strength is typically held constant while the energy is varied. In this Introduction, I have described how the global features in the energy domain cross section $\sigma(E)$ can be extracted by Fourier transforming the cross

section to the time domain. The resulting recurrence spectra $\sigma(T)$ shows peaks at the periods T_k of certain classical orbits of the system. The main difficulty with this approach is that the classical periods depend on the energy $T_k = T_k(E)$. Thus, even though the Fourier transformed spectrum shows peaks at the periods T_k , the peaks are somewhat washed out by the energy varying timescales of the system.

A beautiful alternative to this approach was first introduced by Welge's group and has become known as **scaled variable recurrence spectroscopy**. Here, instead of varying only the energy E while recording the spectrum, both the energy E and magnetic field B are varied, while holding the scaled energy $\epsilon = EB^{-2/3}$ fixed. The resulting photoabsorption cross section $\sigma(w)$ becomes a function of the scaled magnetic field $w = 2\pi B^{-1/3}$. The advantage of this approach is that the classical periods T_k are replaced by the scaled actions \tilde{S}_k of the classical trajectories, which depend only on the scaled energy ϵ . This can be seen in the thorough exploration of the scaling properties of the classical Hamiltonian found in Appendix A.

It is also seen that the Fourier domain of the variable w is the scaled action \tilde{S} . Thus, when the scaled cross section $\sigma(w)$ is Fourier transformed, the resulting scaled recurrence spectrum $\sigma(\tilde{S})$ shows sharp peaks at the scaled actions \tilde{S}_k of the classical orbits. Because the scaled actions themselves do not depend on w , a clean Fourier transformation can be obtained and detailed studies of the scaled action domain physics can be performed. In addition, only a single set of classical orbits (a single value of ϵ) needs to be considered when interpreting or calculating the recurrence spectrum $\sigma(\tilde{S})$.

Other than in the first few experiments on atoms in magnetic fields these scaled variables have been used almost exclusively rather than the physical energy and magnetic field strength (E, B). I follow this usage of scaled variables in this thesis. For the reader unfamiliar with scaled variables I offer a few rules of thumb for thinking about the "scaled" physics of an atomic electron in an external magnetic field. First, the scaled energy ϵ is completely responsible for determining the qualitative features of the electron's motion. As ϵ increases from $-\infty$ to zero, both the classical and quantum Hamiltonians go from being fully integrable (only a Coulomb potential) to being strongly nonintegrable in two dimensions (Coulomb + magnetic field). Second, the scaled magnetic field w functions as the "energy"-like variable when the photoabsorption cross section is measured. The reader is encouraged to ignore the fact that

w is called the “scaled magnetic field” when attempting to gain insight about the qualitative meaning of the scaled variables. Third, the scaled action \tilde{S} becomes the “time”-like variable used to analyze the recurrence spectrum. These general ideas should ease the transition to thinking in terms of scaled variables. Appendix A can be consulted for a more technical discussion of scaled variables.