Appendix C

Semiclassical Green's function amplitude

In Ch. 5, the semiclassical Green's function of Gutzwiller [20, 21, 22] was used to calculate the semiclassical approximation to the long range S-matrix. This Green's function involves a two dimensional Jacobian determinant that measures the stability of the classical orbits contributing to the Green's function. Many of the properties and manipulations of this stability amplitude are subtle and difficult. With this in mind, this Appendix describes some of the simplifications and manipulations of the semiclassical amplitude used in this thesis. For more details on the properties of the semiclassical Green's function, the texts of Brack and Bhaduri [24] and also of Reichl [23] are invaluable. An article by Littlejohn [143] gives a complete, but more mathematical, analysis of the amplitudes described here.

The amplitude of the two-dimensional, semiclassical Green's function, Eq. (5.17), is:

$$|D| = \left|\frac{\partial(p'_\rho, p'_z, t)}{\partial(\rho, z, E)}\right| = \begin{vmatrix} \frac{\partial p'_\rho}{\partial \rho} & \frac{\partial p'_\rho}{\partial z} & \frac{\partial p'_\rho}{\partial E} \\ \frac{\partial p'_z}{\partial \rho} & \frac{\partial p'_z}{\partial z} & \frac{\partial p'_z}{\partial E} \\ \frac{\partial t}{\partial \rho} & \frac{\partial t}{\partial z} & \frac{\partial t}{\partial E} \end{vmatrix}. \tag{C.1}$$

Here and throughout this Appendix, the vertical bars $||$ denote the absolute value of the determinant. The coordinates (ρ, z) and conjugate momenta (p_ρ, p_z) are the standard cylindrical coordinates. It can be shown that the two-by-two subdeterminant in the upper left hand corner of Eq. (C.1) vanishes [24]:

$$\begin{vmatrix} \frac{\partial p'_\rho}{\partial \rho} & \frac{\partial p'_\rho}{\partial z} \\ \frac{\partial p'_z}{\partial \rho} & \frac{\partial p'_z}{\partial z} \end{vmatrix} = 0. \tag{C.2}$$

This is a general property of the semiclassical amplitude that holds in any coordinate system as a consequence of energy conservation. To take this vanishing subdeterminant into account, the derivative $\partial t/\partial E$

in Eq. (C.1) can be set to zero (although it is finite) without consequence.

In addition, a more symmetric form of Eq. (C.1) can be achieved by using the relations:

$$\frac{\partial p'_\rho}{\partial E} = -\frac{\partial^2 S}{\partial E \partial \rho'} = -\frac{\partial t}{\partial \rho'} = -\frac{1}{\dot{\rho}'}, \quad \text{(C.3)}$$

$$\frac{\partial p'_z}{\partial E} = -\frac{\partial^2 S}{\partial E \partial z'} = -\frac{\partial t}{\partial z'} = -\frac{1}{\dot{z}'}. \quad \text{(C.4)}$$

Then the amplitude $|D|$, Eq. (C.1), reads:

$$|D| = \begin{vmatrix} \frac{\partial p'_\rho}{\partial \rho} & \frac{\partial p'_\rho}{\partial z} & -\frac{1}{\dot{\rho}'} \\ \frac{\partial p'_z}{\partial \rho} & \frac{\partial p'_z}{\partial z} & -\frac{1}{\dot{z}'} \\ \frac{1}{\dot{\rho}} & \frac{1}{\dot{z}} & 0 \end{vmatrix} = \left| -\frac{1}{\dot{\rho}} \left(\frac{\partial p'_\rho}{\partial z} \frac{1}{\dot{z}'} - \frac{\partial p'_z}{\partial z} \frac{1}{\dot{\rho}'} \right) + \frac{1}{\dot{z}} \left(\frac{\partial p'_\rho}{\partial \rho} \frac{1}{\dot{z}'} - \frac{\partial p'_z}{\partial \rho} \frac{1}{\dot{\rho}'} \right) \right| \quad \text{(C.5)}$$

Littlejohn [143] has shown that this derivative can be simplified further. This final simplification leads to the form of the amplitude used in Eq. (5.19) to derive the long range S-matrix of Ch. 5. Using energy conservation and various partial derivatives of the Hamiltonian, Littlejohn simplifies the amplitude $|D|$ of Eq. (C.5) into a form that involves the velocity in one direction and a simple partial derivative in the other direction. Two such forms can be derived, corresponding to separating out the velocity in the ρ direction,

$$|D| = \left| \frac{1}{\dot{\rho}} \frac{1}{\dot{\rho}'} \right| \left| \frac{\partial p'_z}{\partial z} \right|_{z'}, \quad \text{(C.6)}$$

or in the z direction,

$$|D| = \left| \frac{1}{\dot{z}} \frac{1}{\dot{z}'} \right| \left| \frac{\partial p'_\rho}{\partial \rho} \right|_{\rho'}. \quad \text{(C.7)}$$

Although it is not obvious, these two forms of the amplitude $|D|$ are identical. In fact, Littlejohn shows that the velocity in **any** direction can be separated out. Thus when spherical polar coordinates (r, θ) are used, the semiclassical amplitude takes the form:

$$|D| = \left| \frac{1}{\dot{r}\dot{r}'} \right| \left| \frac{\partial p'_\theta}{\partial \theta} \right|_{\theta'} \quad \text{(C.8)}$$

This form (C.8) of the amplitude appears in the semiclassical Green's function, Eq. (5.19), used to derive the semiclassical approximation to $\underline{S}^{\mathrm{LR}}$.