Appendix B

ASH CODE EQUATIONS — DETAILED DERIVATIONS

This appendix supplements Chapter 4 by providing detailed derivations of the equations presented therein. In the following section, we perform a formal scale analysis on the fully compressible fluid equations in order to determine the anelastic momentum and energy equations listed in §4.2.3. We then in §B.2 derive the energy conservation equations given in §4.2.4. Lastly, in §B.3 we incorporate the streamfunction formalism presented in §4.2.5 into the anelastic evolution equations, replacing them with the evolution equations listed in §4.2.5.

B.1 ANELASTIC FLUID EQUATIONS

We now apply the scaling outlined in §4.2.3 to the fully compressible fluid equations (4.2)–(4.4) to obtain the set of anelastic equations (4.23)–(4.33), where each of the boxed equations in §B.1.2–§B.1.4 which follow correspond to one of the anelastic equations listed in §4.2.3. The derivations presented here follow Gilman & Glatzmaier (1981).

B.1.1 Order in ϵ of All Dependent Variables

As stated in §4.2.3, we express the four thermodynamic state variables as the sum of a spherically symmetric mean quantity and a fluctuating quantity, as in equation (4.21) of §4.2.3:

$$\begin{aligned} p_{\text{total}}(r,\theta,\phi,t) &= \hat{p}(r) + p(r,\theta,\phi,t), \\ \rho_{\text{total}}(r,\theta,\phi,t) &= \hat{\rho}(r) + \rho(r,\theta,\phi,t), \\ T_{\text{total}}(r,\theta,\phi,t) &= \hat{T}(r) + T(r,\theta,\phi,t), \\ s_{\text{total}}(r,\theta,\phi,t) &= \hat{s}(r) + s(r,\theta,\phi,t). \end{aligned} \tag{B.1}$$

Note that in these equations we have altered the notation used in equation (4.21) by removing the primes from all fluctuating quantities.

The perturbations to the state variables result from convective motions driven by the superadiabatic stratification of the layer, which is characterized by the parameter ϵ defined in equation (4.12). We therefore assume for $\frac{f}{\hat{f}}$ is of order ϵ for any quantity f in equation (B.1), with the anelastic approximation valid when $\epsilon \ll 1$. To filter out sound waves, we assume

$$\frac{\partial \rho}{\partial t} = 0, \tag{B.2}$$

as in equation (4.22) of §4.2.3.

Before performing the scale separation on the compressible fluid equations, we must first determine the order in ϵ of all variables and operators present. We examine variables other than the thermodynamic state variables in the remainder of this section. A summary of these results is provided in Table B.1.

It is important to note that the fluid velocity $\boldsymbol{u}$ is of order $\epsilon^{\frac{1}{2}}$, as suggested by equation (4.14). The main reason for this scaling is that since the kinetic energy of the motions is extracted from the stratification, we have $\frac{\hat{\rho}\boldsymbol{u}^2}{2} \sim \epsilon$ and therefore $\boldsymbol{u} \sim \epsilon^{\frac{1}{2}}$. Given a representative length scale λ_p and velocity $\boldsymbol{u}$, the time scale dt on which the convection modifies the stratification is thus on the order of $\Delta t = \frac{\lambda_p}{|\boldsymbol{u}|} \sim \epsilon^{-\frac{1}{2}}$, causing time derivatives to scale as $\frac{\partial}{\partial t} \sim \epsilon^{\frac{1}{2}}$.

The heuristic arguments presented above also apply to the diffusivity coefficients. Since we expect the eddy dissipation scale to be set by the scale of the convection, we

Table B.1: Order in ϵ for all quantities and operators appearing in the fully compressible fluid equations.

Order in ϵ	Quantities	Operators
ϵ^0	$\hat{p}$, $\hat{\rho}$, $\hat{T}$, $\hat{s}$, g, $\underline{\underline{\boldsymbol{\delta}}}$	$\boldsymbol{\nabla}$, $\boldsymbol{\nabla}\cdot$, $\boldsymbol{\nabla}\times$
$\epsilon^{\frac{1}{2}}$	$\boldsymbol{u}$, ν_{eff}, κ_s, $\boldsymbol{\Omega}$, $\underline{\underline{\boldsymbol{e}}}$	$\dfrac{\partial}{\partial t}$
ϵ^1	p, ρ, T, s	(none)

find that $\nu_{\text{eff}} \approx \lambda_p \boldsymbol{u}$, and thus $\nu_{\text{eff}} \sim \epsilon^{\frac{1}{2}}$ since $\boldsymbol{u} \sim \epsilon^{\frac{1}{2}}$. Similar scaling arguments also hold for κ_r and κ_s.

In the following subsections, we substitute the scaling given in equation (B.1) into the compressible fluid equations and derive the anelastic equations. As stated in §4.2.3, the diffusivities ν_{eff}, κ_r, and κ_s are assumed to be functions of $\hat{\rho}$ only, while the parameters γ and c_p are assumed constant throughout the domain.

B.1.2 Derivation of the Anelastic Mass Continuity Equation

By substituting equations (B.1) into the mass continuity equation (4.2), we obtain

$$\underbrace{\frac{\partial \rho}{\partial t}}_{\text{vanishes}} + \underbrace{\boldsymbol{\nabla} \cdot (\hat{\rho} \boldsymbol{u})}_{\sim \epsilon^{\frac{1}{2}}} + \underbrace{\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u})}_{\sim \epsilon^{\frac{3}{2}}} = 0. \tag{B.3}$$

As indicated, the time derivative of ρ vanishes by equation (B.2). Retaining the highest-ordered term yields the anelastic mass continuity equation,

$$\boxed{\boldsymbol{\nabla} \cdot (\hat{\rho} \boldsymbol{u}) = 0.} \tag{B.4}$$

B.1.3 Derivation of the Anelastic Momentum Equations

By substituting equations (B.1) into the momentum equation (4.3), and using the definition of $\underline{\underline{\boldsymbol{\mathcal{D}}}}$, equation (4.5), we obtain

$$(\hat{\rho}+\rho)\left[\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right] = -\boldsymbol{\nabla}(\hat{p}+p)-(\hat{\rho}+\rho)g\,\hat{\boldsymbol{r}}+2(\hat{\rho}+\rho)(\boldsymbol{u}\times\boldsymbol{\Omega}) + \boldsymbol{\nabla}\cdot\left\{2(\hat{\rho}+\rho)\nu_{\mathrm{eff}}\left[\underline{\underline{\boldsymbol{e}}}-\frac{1}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\,\underline{\underline{\boldsymbol{\delta}}}\right]\right\}. \tag{B.5}$$

After expanding and grouping according to their order in ϵ, we obtain

$$\underbrace{\hat{\rho}\frac{\partial \boldsymbol{u}}{\partial t}+\hat{\rho}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}}_{\sim\epsilon^1}+\underbrace{\rho\frac{\partial \boldsymbol{u}}{\partial t}+\rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}}_{\sim\epsilon^2}+\underbrace{\boldsymbol{\nabla}\hat{p}+\hat{\rho}g\,\hat{\boldsymbol{r}}}_{\sim\epsilon^0}$$
$$=\underbrace{2\hat{\rho}(\boldsymbol{u}\times\boldsymbol{\Omega})-\boldsymbol{\nabla}p-\rho g\,\hat{\boldsymbol{r}}+\boldsymbol{\nabla}\cdot\left\{2\hat{\rho}\nu_{\mathrm{eff}}\left[\underline{\underline{\boldsymbol{e}}}-\frac{1}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\,\underline{\underline{\boldsymbol{\delta}}}\right]\right\}}_{\sim\epsilon^1} \tag{B.6}$$
$$+\underbrace{2\rho(\boldsymbol{u}\times\boldsymbol{\Omega})+\boldsymbol{\nabla}\cdot\left\{2\rho\nu_{\mathrm{eff}}\left[\underline{\underline{\boldsymbol{e}}}-\frac{1}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\,\underline{\underline{\boldsymbol{\delta}}}\right]\right\}}_{\sim\epsilon^2}.$$

The mean momentum equation consists of the terms scaling as ϵ^0, of which only the $\hat{\boldsymbol{r}}$-component remains,

$$\boxed{\frac{\partial \hat{p}}{\partial r}+\hat{\rho}g=0.} \tag{B.7}$$

The first-order (ϵ^1) terms give the fluctuating momentum equation,

$$\boxed{\hat{\rho}\frac{\partial \boldsymbol{u}}{\partial t}=2\hat{\rho}(\boldsymbol{u}\times\boldsymbol{\Omega})-\hat{\rho}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}-\boldsymbol{\nabla}p-\rho g\,\hat{\boldsymbol{r}}+\boldsymbol{\nabla}\cdot\hat{\underline{\underline{\boldsymbol{\mathcal{D}}}}},} \tag{B.8}$$

where the anelastic viscous stress tensor $\hat{\underline{\underline{\boldsymbol{\mathcal{D}}}}}$ is defined

$$\boxed{\hat{\underline{\underline{\boldsymbol{\mathcal{D}}}}}=2\hat{\rho}\nu_{\mathrm{eff}}\left[\underline{\underline{\boldsymbol{e}}}-\frac{1}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\,\underline{\underline{\boldsymbol{\delta}}}\right].} \tag{B.9}$$

B.1.4 Derivation of the Anelastic Energy Equation

By substituting equations (B.1) into the energy equation (4.4), and using the definition of Φ, equation (4.6), we obtain

$$(\hat{\rho}+\rho)(\hat{T}+T)\left[\frac{\partial(\hat{s}+s)}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})(\hat{s}+s)\right] = -\boldsymbol{\nabla}\cdot\hat{\boldsymbol{q}}_{\mathrm{eff}}+2(\hat{\rho}+\rho)\left[\underline{\underline{\boldsymbol{e}}}:\underline{\underline{\boldsymbol{e}}}-\frac{1}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{u})^2\right], \tag{B.10}$$

where the anelastic heat flux $\hat{\boldsymbol{q}}_{\text{eff}}$ is defined

$$\boxed{\hat{\boldsymbol{q}}_{\text{eff}} = -\kappa_r \hat{\rho} c_p \boldsymbol{\nabla}(\hat{T} + T) - \kappa_s \hat{\rho} \hat{T} \boldsymbol{\nabla}(\hat{s} + s).} \tag{B.11}$$

In the above equation, we have explicitly included the contributions to the diffusive transport of internal energy by the fluctuating temperature and entropy gradients. Similarly, we also keep the fluctuating entropy gradient in the advection term, so that the anelastic energy equation becomes

$$\boxed{\hat{\rho}\hat{T}\frac{\partial s}{\partial t} = -\boldsymbol{\nabla} \cdot \hat{\boldsymbol{q}}_{\text{eff}} - \hat{\rho}\hat{T}(\boldsymbol{u} \cdot \boldsymbol{\nabla})(\hat{s} + s) + \hat{\Phi},} \tag{B.12}$$

where the anelastic viscous heating term $\hat{\Phi}$ is defined

$$\hat{\Phi} = 2\hat{\rho}\nu_{\text{eff}} \left[\underline{\underline{\boldsymbol{e}}} : \underline{\underline{\boldsymbol{e}}} - \frac{1}{3}(\boldsymbol{\nabla} \cdot \boldsymbol{u})^2 \right]. \tag{B.13}$$

Note that we have used the anelastic stress tensor $\underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}$ as defined in equation (B.9).

B.1.5 Derivation of the Anelastic Equations of State

Substituting equations (B.1) into the equation of state (4.10) yields

$$\hat{p} + p = \frac{\gamma - 1}{\gamma} c_p (\hat{\rho} + \rho)(\hat{T} + T). \tag{B.14}$$

Collecting the zeroth-order terms gives

$$\boxed{\hat{p} = \frac{\gamma - 1}{\gamma} c_p \hat{\rho} \hat{T},} \tag{B.15}$$

while the first-order terms are

$$p = \frac{\gamma - 1}{\gamma} c_p \left(\rho \hat{T} + \hat{\rho} T \right), \tag{B.16}$$

or after dividing by equation (B.15) we arrive at

$$\boxed{\frac{p}{\hat{p}} = \frac{\rho}{\hat{\rho}} + \frac{T}{\hat{T}}.} \tag{B.17}$$

Substituting equations (B.1) into the entropy equation (4.11) yields

$$\hat{s} + s = c_p \left[\frac{1}{\gamma} \ln(\hat{p} + p) - \ln(\hat{\rho} + \rho) \right], \tag{B.18}$$

$$= c_p \left[\frac{1}{\gamma} \left(\ln \hat{p} + \ln \left[1 + \frac{p}{\hat{p}} \right] \right) - \left(\ln \hat{\rho} + \ln \left[1 + \frac{\rho}{\hat{\rho}} \right] \right) \right]. \tag{B.19}$$

Since $\left| \frac{p}{\hat{p}} \right| \ll 1$ and $\left| \frac{\rho}{\hat{\rho}} \right| \ll 1$, we can use the series representation of $\ln(1 + x)$,

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, \qquad \text{valid for } -1 < x \leq 1. \tag{B.20}$$

Keeping only the highest-order term in the series, we obtain

$$\hat{s} + s = c_p \left[\frac{1}{\gamma} \left(\ln \hat{p} + \frac{p}{\hat{p}} \right) - \left(\ln \hat{\rho} + \frac{\rho}{\hat{\rho}} \right) \right]. \tag{B.21}$$

The zeroth-order terms are

$$\hat{s} = c_p \left(\frac{1}{\gamma} \ln \hat{p} - \ln \hat{\rho} \right), \tag{B.22}$$

which after taking a radial derivative become

$$\boxed{\frac{d\hat{s}}{dr} = c_p \left(\frac{1}{\gamma \hat{p}} \frac{d\hat{p}}{dr} - \frac{1}{\hat{\rho}} \frac{d\hat{\rho}}{dr} \right),} \tag{B.23}$$

while the first-order terms give

$$\boxed{\frac{s}{c_p} = \frac{p}{\gamma \hat{p}} - \frac{\rho}{\hat{\rho}}.} \tag{B.24}$$

B.2 ANELASTIC EQUATION ENERGETICS

B.2.1 Derivation of the Kinetic Energy Conservation Equation

The equation describing the conservation of kinetic energy density $\mathcal{E}_k = \frac{\hat{\rho} \boldsymbol{u} \cdot \boldsymbol{u}}{2}$ is formed by taking $\boldsymbol{u} \cdot$ each term in the anelastic momentum equation (B.8),

$$\hat{\rho} \frac{\partial \boldsymbol{u}}{\partial t} = 2\hat{\rho}(\boldsymbol{u} \times \boldsymbol{\Omega}) - \hat{\rho}(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} - \boldsymbol{\nabla} p - \rho g \,\hat{\boldsymbol{r}} + \boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}. \tag{B.25}$$

Starting with the time-derivative term, we have

$$\hat{\rho} \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\hat{\rho} \boldsymbol{u} \cdot \boldsymbol{u}}{2} \right) = \frac{\partial \mathcal{E}_k}{\partial t}, \tag{B.26}$$

where we have made use of the fact that $\dfrac{\partial \hat{\rho}}{\partial t} = 0$.

The Coriolis term,

$$2\hat{\rho}\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{\Omega}) = 0, \tag{B.27}$$

vanishes since it acts perpendicular to the motion and thus cannot perform any work.

The inertial term is

$$\hat{\rho}\boldsymbol{u} \cdot [(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}] = \boldsymbol{\nabla} \cdot \left[\boldsymbol{u}\left(\frac{\hat{\rho}\boldsymbol{u} \cdot \boldsymbol{u}}{2}\right)\right]. \tag{B.28}$$

This relation is most easily verified by using the following vector identity,

$$\boldsymbol{\nabla} \cdot (f\boldsymbol{A}) = \boldsymbol{A} \cdot (\boldsymbol{\nabla} f) + f(\boldsymbol{\nabla} \cdot \boldsymbol{A}), \tag{B.29}$$

to expand $\boldsymbol{\nabla} \cdot \left[\boldsymbol{u}\left(\frac{\hat{\rho}\boldsymbol{u} \cdot \boldsymbol{u}}{2}\right)\right] = \boldsymbol{\nabla} \cdot \left[\left(\frac{\hat{\rho}\boldsymbol{u}}{2}\right)\boldsymbol{u} \cdot \boldsymbol{u}\right]$, and showing that it equals $\hat{\rho}\boldsymbol{u} \cdot [(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}]$:

$$\boldsymbol{\nabla} \cdot \left[\left(\frac{\hat{\rho}\boldsymbol{u}}{2}\right)\boldsymbol{u} \cdot \boldsymbol{u}\right] = \frac{\hat{\rho}\boldsymbol{u}}{2} \cdot [\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{u})] + \underbrace{\frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{u})\boldsymbol{\nabla} \cdot (\hat{\rho}\boldsymbol{u})}_{\text{vanishes by equation (B.4)}} \tag{B.30}$$

$$= \underbrace{\hat{\rho}\boldsymbol{u} \cdot [\boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{u})]}_{\text{vanishes}} + \hat{\rho}\boldsymbol{u} \cdot [(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}]. \tag{B.31}$$

Note that this derivation shows that the inertial term can be written as the divergence of the kinetic energy flux.

The remaining terms are simply

$$\boldsymbol{u} \cdot \boldsymbol{\nabla} p + \rho g \boldsymbol{u} \cdot \hat{\boldsymbol{r}} + \boldsymbol{u} \cdot \left(\boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right). \tag{B.32}$$

Collecting terms, we obtain the kinetic energy conservation equation listed as equation (4.34) of Chapter 4,

$$\boxed{\frac{\partial \mathcal{E}_k}{\partial t} = -\boldsymbol{u} \cdot \boldsymbol{\nabla} p - \rho g \boldsymbol{u} \cdot \hat{\boldsymbol{r}} + \boldsymbol{u} \cdot \left(\boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right) - \boldsymbol{\nabla} \cdot \left[\boldsymbol{u}\left(\frac{\hat{\rho}\boldsymbol{u} \cdot \boldsymbol{u}}{2}\right)\right].} \tag{B.33}$$

B.2.2 Derivation of the Internal Energy Conservation Equation

Since the time derivatives of $\hat{\rho}$ and $\hat{T}$ are zero, the equation describing the time dependence of internal energy density $\mathcal{E}_s = \hat{\rho}\hat{T}s$ is the anelastic internal energy equation (B.12),

$$\frac{\partial \mathcal{E}_s}{\partial t} = \boldsymbol{\nabla} \cdot \hat{\boldsymbol{q}}_{\text{eff}} - \hat{\rho}\hat{T}(\boldsymbol{u} \cdot \boldsymbol{\nabla})(\hat{s} + s) + \hat{\Phi}, \tag{B.34}$$

where the time derivative term $\hat{\rho}\hat{T}\frac{\partial s}{\partial t}$ has been replaced by $\frac{\partial}{\partial t}(\hat{\rho}\hat{T}s) = \frac{\partial \mathcal{E}_s}{\partial t}$.

Additional insight may be gained by rewriting the inertial term as follows:

$$\begin{aligned}
&- \hat{\rho}\hat{T}(\boldsymbol{u} \cdot \boldsymbol{\nabla})s \\
&\quad = -c_p\hat{\rho}\hat{T}(\boldsymbol{u} \cdot \boldsymbol{\nabla})\left[\frac{T}{\hat{T}} - \frac{\gamma - 1}{\gamma}\frac{p}{\hat{p}}\right] \\
&\quad = -c_p\hat{\rho}\hat{T}\boldsymbol{u} \cdot \left[\frac{\boldsymbol{\nabla}T}{\hat{T}} - \frac{\gamma - 1}{\gamma}\frac{\boldsymbol{\nabla}p}{\hat{p}} - \frac{T}{\hat{T}^2}\frac{d\hat{T}}{dr}\hat{\boldsymbol{r}} + \frac{\gamma - 1}{\gamma}\frac{p}{\hat{p}^2}\frac{d\hat{p}}{dr}\hat{\boldsymbol{r}}\right] \\
&\quad = -c_p\hat{\rho}\boldsymbol{u} \cdot \boldsymbol{\nabla}T + \boldsymbol{u} \cdot \boldsymbol{\nabla}p + c_p u_r \frac{\hat{\rho}T}{\hat{T}}\frac{d\hat{T}}{dr} - u_r\frac{p}{\hat{p}}\frac{d\hat{p}}{dr} \\
&\quad = -c_p\boldsymbol{\nabla} \cdot (\hat{\rho}T\boldsymbol{u}) + \boldsymbol{u} \cdot \boldsymbol{\nabla}p + c_p u_r\hat{\rho}T\left[\frac{1}{c_p}\frac{d\hat{s}}{dr} + \frac{\gamma - 1}{\gamma\hat{p}}\frac{d\hat{p}}{dr}\right] - u_r\frac{p}{\hat{p}}\frac{d\hat{p}}{dr} \\
&\quad = -c_p\boldsymbol{\nabla} \cdot (\hat{\rho}T\boldsymbol{u}) + \boldsymbol{u} \cdot \boldsymbol{\nabla}p + u_r\hat{\rho}T\frac{d\hat{s}}{dr} + u_r\left[\frac{T}{\hat{T}} - \frac{p}{\hat{p}}\right]\frac{d\hat{p}}{dr} \\
&\quad = -c_p\boldsymbol{\nabla} \cdot (\hat{\rho}T\boldsymbol{u}) + \boldsymbol{u} \cdot \boldsymbol{\nabla}p - u_r\frac{\rho}{\hat{\rho}}\frac{d\hat{p}}{dr} + \underbrace{u_r\hat{\rho}T\frac{d\hat{s}}{dr}}_{\approx 0,\ \text{higher order}} \\
&\quad = -c_p\boldsymbol{\nabla} \cdot (\hat{\rho}T\boldsymbol{u}) + \boldsymbol{u} \cdot \boldsymbol{\nabla}p + u_r\rho g
\end{aligned}$$

where $u_r = \boldsymbol{u} \cdot \hat{\boldsymbol{r}}$, and where the mean equations (B.7), (B.15), and (B.23) as well as the dynamic equations of state (B.17) and (B.24) have been used throughout. Using this form of the inertial term, we obtain the internal energy conservation equation listed as equation (4.35) of Chapter 4,

$$\boxed{\frac{\partial \mathcal{E}_s}{\partial t} = \boldsymbol{\nabla} \cdot \hat{\boldsymbol{q}}_{\text{eff}} + \boldsymbol{u} \cdot \boldsymbol{\nabla}p + \rho g\boldsymbol{u} \cdot \hat{\boldsymbol{r}} - \boldsymbol{\nabla} \cdot (c_p\hat{\rho}T\boldsymbol{u}) + \hat{\Phi}.} \tag{B.35}$$

B.2.3 Derivation of the Total Energy Conservation Equation

Adding together the energy conservation equations (B.33) and (B.35) yields the total energy equation (4.36) of Chapter 4,

$$\boxed{\frac{\partial(\mathcal{E}_k + \mathcal{E}_s)}{\partial t} = \boldsymbol{\nabla}\cdot\left(\boldsymbol{u}\cdot\hat{\underline{\underline{\boldsymbol{\mathcal{D}}}}}\right) + \boldsymbol{\nabla}\cdot\left[\boldsymbol{u}\left(\frac{\hat{\rho}\boldsymbol{u}\cdot\boldsymbol{u}}{2}\right)\right] + \boldsymbol{\nabla}\cdot\hat{\boldsymbol{q}}_{\text{eff}} - \boldsymbol{\nabla}\cdot(c_p\hat{\rho}T\boldsymbol{u}).} \tag{B.36}$$

In the above equation, the viscous energy source term $\boldsymbol{\nabla}\cdot\left(\boldsymbol{u}\cdot\hat{\underline{\underline{\boldsymbol{\mathcal{D}}}}}\right)$ is arrived at by combining the diffusion terms in equations (B.33) and (B.35), which we now show using indicial notation in Cartesian coordinates.

$$\boldsymbol{u}\cdot\left(\boldsymbol{\nabla}\cdot\hat{\underline{\underline{\boldsymbol{\mathcal{D}}}}}\right) + \hat{\Phi} = u_i\frac{\partial\mathcal{D}_{ij}}{\partial x_j} + \mathcal{D}_{ij}\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}(u_i\mathcal{D}_{ij}) = \boldsymbol{\nabla}\cdot\left(\boldsymbol{u}\cdot\hat{\underline{\underline{\boldsymbol{\mathcal{D}}}}}\right), \tag{B.37}$$

where the we have used the equality

$$\mathcal{D}_{ij}\frac{\partial u_i}{\partial x_j} = 2\hat{\rho}\nu_{\text{eff}}\left[e_{ij}\frac{\partial u_i}{\partial x_j} - \frac{1}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\frac{\partial u_i}{\partial x_j}\delta_{ij}\right] \tag{B.38}$$

$$= 2\hat{\rho}\nu_{\text{eff}}\left[e_{ij}e_{ij} - \frac{1}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{u})^2\right] \qquad \text{since } \frac{\partial u_i}{\partial x_j}\delta_{ij} = \boldsymbol{\nabla}\cdot\boldsymbol{u} \tag{B.39}$$

$$= 2\hat{\rho}\nu_{\text{eff}}\left[\underline{\underline{\boldsymbol{e}}} : \underline{\underline{\boldsymbol{e}}} - \frac{1}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{u})^2\right] \tag{B.40}$$

$$= \hat{\Phi}. \tag{B.41}$$

Going from equation (B.38) to (B.39) is achieved by using the fact that the doubly contracted tensor product of a symmetric tensor and an antisymmetric tensor is zero. Since the strain rate tensor $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is symmetric (as can be seen from its Cartesian representation), the product $\frac{\partial u_i}{\partial x_j}e_{ij}$ is equal to the product of e_{ij} and the symmetric part of $\frac{\partial u_i}{\partial x_j}$, which is simply e_{ij}.

B.3 THE NUMERICAL EVOLUTION EQUATIONS

We now proceed to derive equations (4.38)–(4.40) of Chapter 4.

B.3.1 Streamfunction Formalism

Any vector $\boldsymbol{A}$ for which $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ can be decomposed into poloidal and toroidal streamfunctions W and Z, respectively:

$$\boldsymbol{A} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times W\,\hat{\boldsymbol{r}}) + \boldsymbol{\nabla} \times Z\,\hat{\boldsymbol{r}}, \tag{B.42}$$

such that $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ is identically satisfied at all times. Because

$$\boldsymbol{\nabla} \cdot (\hat{\rho}\boldsymbol{u}) = 0, \tag{B.43}$$

by equation (B.4), we may write

$$\hat{\rho}\boldsymbol{u} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times W\,\hat{\boldsymbol{r}}) + \boldsymbol{\nabla} \times Z\,\hat{\boldsymbol{r}}. \tag{B.44}$$

The scalar quantities W and Z are the poloidal and toroidal streamfunctions for the mass flux, and it is equations describing the evolution of these quantities (along with p and s) that are solved by the ASH code. We outline the derivations of these equations in Table B.2, while presenting detailed derivations in §B.3.4–§B.3.7.

B.3.2 Streamfunction Identities

In this section, we list some identities used in the upcoming sections. Expanding the vector cross products from equation (B.42) in spherical polar coordinates, the three components of $\hat{\rho}\boldsymbol{u}$ are found to be

$$\hat{\rho}\boldsymbol{u} = -\left(\nabla_\perp^2 W\right)\hat{\boldsymbol{r}} + \left[\frac{1}{r}\frac{\partial^2 W}{\partial r \partial \theta} + \frac{1}{r\sin\theta}\frac{\partial Z}{\partial \phi}\right]\hat{\boldsymbol{\theta}} + \left[\frac{1}{r\sin\theta}\frac{\partial^2 W}{\partial r \partial \phi} - \frac{1}{r}\frac{\partial Z}{\partial \theta}\right]\hat{\boldsymbol{\phi}}, \tag{B.45}$$

while the curl of $\hat{\rho}\boldsymbol{u}$ is

$$\begin{aligned}\boldsymbol{\nabla} \times \hat{\rho}\boldsymbol{u} = &-\left(\nabla_\perp^2 Z\right)\hat{\boldsymbol{r}} + \left[-\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{\partial^2}{\partial r^2} + \nabla_\perp^2\right)W + \frac{1}{r}\frac{\partial^2 Z}{\partial r \partial \theta}\right]\hat{\boldsymbol{\theta}} \\ &+ \left[\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial^2}{\partial r^2} + \nabla_\perp^2\right)W + \frac{1}{r\sin\theta}\frac{\partial^2 Z}{\partial r \partial \phi}\right]\hat{\boldsymbol{\phi}}.\end{aligned} \tag{B.46}$$

Table B.2: Strategy employed to obtain the ASH code evolution equations.

Apply this operator	to this equation	to obtain the evolution equation for	derived in	as equation
$\hat{\boldsymbol{r}}\cdot$	(B.8)	$\frac{\partial}{\partial t}(\nabla_\perp^2 W)$	§B.3.4	(B.123)
$\boldsymbol{\nabla}_\perp\cdot$	(B.8)	$\frac{\partial}{\partial t}\left(\nabla_\perp^2 \frac{\partial W}{\partial r}\right)$	§B.3.5	(B.132)
$\hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla}\times$	(B.8)	$\frac{\partial}{\partial t}(\nabla_\perp^2 Z)$	§B.3.6	(B.140)
[none]	(B.12)	$\frac{\partial s}{\partial t}$	§B.3.7	(B.144)

In the above two expressions, we have used the horizontal Laplacian operator,

$$\nabla_\perp^2 = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \tag{B.47}$$

defined such that the total Laplacian operator is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \nabla_\perp^2. \tag{B.48}$$

The horizontal Laplacian operator commutes with radial derivatives in the following manner:

$$\left(\frac{\partial}{\partial r}\nabla_\perp^2\right) f(r,\theta,\phi) = \nabla_\perp^2\left(\frac{\partial}{\partial r} - \frac{2}{r}\right) f(r,\theta,\phi), \tag{B.49}$$

$$\left(\frac{\partial^2}{\partial r^2}\nabla_\perp^2\right) f(r,\theta,\phi) = \nabla_\perp^2\left(\frac{\partial^2}{\partial r^2} - \frac{4}{r}\frac{\partial}{\partial r} + \frac{6}{r^2}\right) f(r,\theta,\phi), \tag{B.50}$$

$$\left(\frac{\partial^3}{\partial r^3}\nabla_\perp^2\right) f(r,\theta,\phi) = \nabla_\perp^2\left(\frac{\partial^3}{\partial r^3} - \frac{6}{r}\frac{\partial^2}{\partial r^2} + \frac{18}{r^2}\frac{\partial}{\partial r} - \frac{24}{r^3}\right) f(r,\theta,\phi). \tag{B.51}$$

B.3.3 Components of the Divergence of the Viscous Stress Tensor

In this section, we evaluate the components of the anelastic viscous stress tensor, $\boldsymbol{\nabla}\cdot\underline{\underline{\hat{\mathcal{D}}}}$, in preparation for their use in §B.3.4–§B.3.7.

B.3.3.1 Preliminaries

The anelastic stress tensor is

$$
\begin{aligned}
\underline{\underline{\hat{\mathcal{D}}}} &= 2\hat{\rho}\nu \left[\underline{\underline{e}} - \frac{1}{3}(\nabla \cdot \boldsymbol{u}) \underline{\underline{\delta}} \right] \qquad \text{by equation (B.9)} \\
&= 2\nu \left[\hat{\rho}\underline{\underline{e}} - \frac{\hat{\rho}}{3}(\nabla \cdot \boldsymbol{u}) \underline{\underline{\delta}} \right] \\
&= 2\nu \left[\hat{\rho}\underline{\underline{e}} - \frac{1}{3}\Big(\nabla \cdot (\hat{\rho}\boldsymbol{u}) - \boldsymbol{u} \cdot \nabla\hat{\rho}\Big) \underline{\underline{\delta}} \right] \\
&= 2\nu \left[\hat{\rho}\underline{\underline{e}} + \frac{\beta}{3}\,\hat{\rho}u_r\, \underline{\underline{\delta}} \right], \qquad \text{since } \nabla \cdot (\hat{\rho}\boldsymbol{u}) = 0 \text{ by equation (B.4)}
\end{aligned} \tag{B.52}
$$

where β is defined below in equation (B.56) and where we have dropped the "eff" subscript from ν_{eff}, hereafter using ν instead. To evaluate $\nabla \cdot \underline{\underline{\hat{\mathcal{D}}}}$, we will need the individual components of the tensor $\hat{\rho}\underline{\underline{e}}$ in spherical polar coordinates,

$$
\begin{aligned}
\hat{\rho}e_{rr} &= \frac{\partial(\hat{\rho}u_r)}{\partial r} - \beta\hat{\rho}u_r \\
\hat{\rho}e_{\theta\theta} &= \frac{1}{r}\frac{\partial(\hat{\rho}u_\theta)}{\partial\theta} + \frac{1}{r}\,\hat{\rho}u_r, \\
\hat{\rho}e_{\phi\phi} &= \frac{1}{r\sin\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi} + \frac{1}{r}\,\hat{\rho}u_r + \frac{\cos\theta}{r\sin\theta}\,\hat{\rho}u_\theta, \\
\hat{\rho}e_{r\theta} &= \frac{1}{2}\frac{\partial(\hat{\rho}u_\theta)}{\partial r} - \frac{\beta}{2}\,\hat{\rho}u_\theta - \frac{1}{2r}\,\hat{\rho}u_\theta + \frac{1}{2r}\frac{\partial(\hat{\rho}u_r)}{\partial\theta}, \\
\hat{\rho}e_{r\phi} &= \frac{1}{2r\sin\theta}\frac{\partial(\hat{\rho}u_r)}{\partial\phi} + \frac{1}{2}\frac{\partial(\hat{\rho}u_\phi)}{\partial r} - \frac{\beta}{2}\,\hat{\rho}u_\phi - \frac{1}{2r}\,\hat{\rho}u_\phi, \\
\hat{\rho}e_{\theta\phi} &= \frac{1}{2r}\frac{\partial(\hat{\rho}u_\phi)}{\partial\theta} - \frac{\cos\theta}{2r\sin\theta}\,\hat{\rho}u_\phi + \frac{1}{2r\sin\theta}\frac{\partial(\hat{\rho}u_\theta)}{\partial\phi},
\end{aligned} \tag{B.53}
$$

where we have again used the definition of β given in equation (B.56) below. It will also prove useful to have handy the expression for $\nabla \cdot (\hat{\rho}\boldsymbol{u}) = 0$,

$$
\frac{1}{r^2}\frac{\partial(r^2\hat{\rho}u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,\hat{\rho}u_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi} = 0, \tag{B.54}
$$

which vanishes by equation (B.4). Finally, we define the following functions of r,

$$
\alpha = \frac{d\ln\nu}{dr} \tag{B.55}
$$

and

$$
\beta = \frac{d\ln\hat{\rho}}{dr}. \tag{B.56}
$$

B.3.3.2 Radial Component

For any tensor $\underline{\underline{\mathcal{T}}}$, the radial component of its divergence is

$$\hat{\boldsymbol{r}} \cdot \left(\boldsymbol{\nabla} \cdot \underline{\underline{\mathcal{T}}}\right) = \underbrace{\frac{1}{r^2}\frac{\partial(r^2 \mathcal{T}_{rr})}{\partial r}}_{\boxed{1}} + \underbrace{\frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, \mathcal{T}_{r\theta})}{\partial \theta}}_{\boxed{2}} + \underbrace{\frac{1}{r\sin\theta}\frac{\partial \mathcal{T}_{r\phi}}{\partial \phi}}_{\boxed{3}} - \underbrace{\frac{\mathcal{T}_{\theta\theta} + \mathcal{T}_{\phi\phi}}{r}}_{\boxed{4}}. \tag{B.57}$$

In this section, we evaluate equation (B.57) for $\underline{\underline{\mathcal{T}}} = \underline{\underline{\hat{\mathcal{D}}}}$ using the definition of $\underline{\underline{\hat{\mathcal{D}}}}$ given in equation (B.52) along with the strain rate tensor components of equation (B.53). Starting with the first term, we have

$$\begin{aligned}
\boxed{1} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\, 2\nu \left(\frac{\partial(\hat{\rho} u_r)}{\partial r} - \frac{2\beta}{3}\hat{\rho} u_r\right)\right] \\
&= \frac{2\nu}{r^2}\left\{\alpha r^2 \left[\frac{\partial(\hat{\rho} u_r)}{\partial r} - \frac{2\beta}{3}\hat{\rho} u_r\right] + \frac{\partial}{\partial r}\left(r^2\left[\frac{\partial(\hat{\rho} u_r)}{\partial r} - \frac{2\beta}{3}\hat{\rho} u_r\right]\right)\right\} \\
&= \underbrace{2\nu\alpha\left[\frac{\partial(\hat{\rho} u_r)}{\partial r} - \frac{2\beta}{3}\hat{\rho} u_r\right]}_{\boxed{1a}} + \underbrace{\frac{2\nu}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial(\hat{\rho} u_r)}{\partial r}\right]}_{\boxed{1b}} - \underbrace{\frac{4\nu\beta}{3r^2}\frac{\partial(r^2\hat{\rho} u_r)}{\partial r}}_{\boxed{1c}} - \underbrace{\frac{4\nu}{3}\frac{d\beta}{dr}\hat{\rho} u_r}_{\boxed{1d}}.
\end{aligned} \tag{B.58}$$

The second term is

$$\begin{aligned}
\boxed{2} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta\, 2\nu \left(\frac{1}{2}\frac{\partial(\hat{\rho} u_\theta)}{\partial r} - \frac{\beta}{2}\,\hat{\rho} u_\theta - \frac{1}{2r}\,\hat{\rho} u_\theta + \frac{1}{2r}\frac{\partial(\hat{\rho} u_r)}{\partial \theta}\right)\right] \\
&= \frac{\nu}{r\sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta \left(\frac{\partial(\hat{\rho} u_\theta)}{\partial r} - \beta\hat{\rho} u_\theta - \frac{1}{r}\,\hat{\rho} u_\theta + \frac{1}{r}\frac{\partial(\hat{\rho} u_r)}{\partial \theta}\right)\right] \\
&= \underbrace{\nu\frac{\partial}{\partial r}\left[\frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,\hat{\rho} u_\theta)}{\partial \theta}\right]}_{\boxed{2a}} - \underbrace{\frac{\nu\beta}{r\sin\theta}\frac{\partial(\sin\theta\,\hat{\rho} u_\theta)}{\partial \theta}}_{\boxed{2b}} + \underbrace{\frac{\nu}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial(\hat{\rho} u_r)}{\partial \theta}\right]}_{\boxed{2c}},
\end{aligned} \tag{B.59}$$

while the third term is

$$\begin{aligned}
\boxed{3} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left[2\nu \left(\frac{1}{2r\sin\theta}\frac{\partial(\hat{\rho} u_r)}{\partial \phi} + \frac{1}{2}\frac{\partial(\hat{\rho} u_\phi)}{\partial r} - \frac{\beta}{2}\,\hat{\rho} u_\phi - \frac{1}{2r}\,\hat{\rho} u_\phi\right)\right] \\
&= \frac{\nu}{r\sin\theta}\frac{\partial}{\partial \phi}\left[\frac{1}{r\sin\theta}\frac{\partial(\hat{\rho} u_r)}{\partial \phi} + \frac{\partial(\hat{\rho} u_\phi)}{\partial r} - \beta\hat{\rho} u_\phi - \frac{1}{r}\,\hat{\rho} u_\phi\right] \\
&= \underbrace{\nu\frac{\partial}{\partial r}\left[\frac{1}{r\sin\theta}\frac{\partial(\hat{\rho} u_\phi)}{\partial \phi}\right]}_{\boxed{3a}} + \underbrace{\frac{\nu}{r^2\sin^2\theta}\frac{\partial^2(\hat{\rho} u_r)}{\partial \phi^2}}_{\boxed{3b}} - \underbrace{\frac{\nu\beta}{r\sin\theta}\frac{\partial(\hat{\rho} u_\phi)}{\partial \phi}}_{\boxed{3c}}.
\end{aligned} \tag{B.60}$$

The last term is

$$\begin{aligned}
\boxed{4} &= -\frac{2\nu}{r}\left[\frac{1}{r}\frac{\partial(\hat{\rho}u_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi} + \frac{2}{r}\hat{\rho}u_r + \frac{\cos\theta}{r\sin\theta}\,\hat{\rho}u_\theta + \frac{2\beta}{3}\hat{\rho}u_r\right] \\
&= -\frac{2\nu}{r}\left[\frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,\hat{\rho}u_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi}\right] - \frac{4\nu}{r^2}\,\hat{\rho}u_r - \frac{4\nu\beta}{3r}\hat{\rho}u_r \\
&= \frac{2\nu}{r}\left[\frac{1}{r^2}\frac{\partial(r^2\hat{\rho}u_r)}{\partial r}\right] - \frac{4\nu}{r^2}\,\hat{\rho}u_r - \frac{4\nu\beta}{3r}\hat{\rho}u_r \qquad \text{by equation (B.54)} \\
&= \frac{2\nu}{r}\left[\frac{\partial(\hat{\rho}u_r)}{\partial r} + \frac{2}{r}\hat{\rho}u_r\right] - \frac{4\nu}{r^2}\hat{\rho}u_r - \frac{4\nu\beta}{3r}\hat{\rho}u_r \\
&= \underbrace{\frac{2\nu}{r}\frac{\partial(\hat{\rho}u_r)}{\partial r}}_{\boxed{4a}} - \underbrace{\frac{4\nu\beta}{3r}\hat{\rho}u_r}_{\boxed{4b}}\,.
\end{aligned} \tag{B.61}$$

To evaluate $\hat{\boldsymbol{r}} \cdot \left[\boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right] = \boxed{1} + \boxed{2} + \boxed{3} + \boxed{4}$ by equation (B.57), we combine pieces from equations (B.58)–(B.61) as indicated in the following expressions:

$$\begin{aligned}
\boxed{1c} + \boxed{2b} + \boxed{3c} + \boxed{4b} &= -\nu\beta\left[\frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,\hat{\rho}u_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi}\right] + \boxed{1c} + \boxed{4b} \\
&= \nu\beta\left[\frac{1}{r^2}\frac{\partial(r^2\hat{\rho}u_r)}{\partial r}\right] + \boxed{1c} + \boxed{4b} \qquad \text{by equation (B.54)} \\
&= -\frac{\nu\beta}{3r^2}\frac{\partial(r^2\hat{\rho}u_r)}{\partial r} + \boxed{4b} \\
&= -\frac{\nu\beta}{3}\frac{\partial(\hat{\rho}u_r)}{\partial r} - \frac{2\nu\beta}{3r}\hat{\rho}u_r + \boxed{4b} \\
&= -\frac{\nu\beta}{3}\frac{\partial(\hat{\rho}u_r)}{\partial r} - \frac{2\nu\beta}{r}\hat{\rho}u_r,
\end{aligned} \tag{B.62}$$

$$\begin{aligned}
\boxed{1b} + \boxed{2a} + \boxed{3a} &= \nu\frac{\partial}{\partial r}\left[\frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,\hat{\rho}u_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi}\right] + \boxed{1b} \\
&= -\nu\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial(r^2\hat{\rho}u_r)}{\partial r}\right] + \boxed{1b} \qquad \text{by equation (B.54)} \\
&= \nu\left[\frac{\partial^2(\hat{\rho}u_r)}{\partial r^2} + \frac{2}{r}\frac{\partial(\hat{\rho}u_r)}{\partial r} + \frac{2}{r^2}\hat{\rho}u_r\right],
\end{aligned} \tag{B.63}$$

$$\boxed{2c} + \boxed{3b} = \nu\nabla_\perp^2(\hat{\rho}u_r) \qquad \text{by equation (B.47).} \tag{B.64}$$

The remaining terms are unchanged:

$$\boxed{1a} + \boxed{1d} + \boxed{4a} = 2\nu\alpha\left[\frac{\partial(\hat{\rho}u_r)}{\partial r} - \frac{2\beta}{3}\hat{\rho}u_r\right] - \frac{4\nu}{3}\frac{d\beta}{dr}\hat{\rho}u_r + \frac{2\nu}{r}\frac{\partial(\hat{\rho}u_r)}{\partial r}. \tag{B.65}$$

Combining equations (B.62)–(B.65) and grouping by derivatives of $\hat{\rho} u_r$, we obtain

$$\hat{\boldsymbol{r}} \cdot \left[\boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right] = \nu \left\{ \frac{\partial^2 (\hat{\rho} u_r)}{\partial r^2} + \left[2\alpha - \frac{\beta}{3} + \frac{4}{r}\right] \frac{\partial (\hat{\rho} u_r)}{\partial r} + \left[\nabla_\perp^2 - \frac{4\alpha\beta}{3} - \frac{4}{3}\frac{d\beta}{dr} - \frac{2\beta}{r} + \frac{2}{r^2}\right] \hat{\rho} u_r \right\}. \tag{B.66}$$

The last step is to eliminate the radial mass flux $\hat{\rho} u_r$ in favor of the streamfunction W by substituting $\hat{\rho} u_r = -\nabla_\perp^2 W$, by equation (B.45). Using the commutation identities of equations (B.49) and (B.50), equation (B.66) thus becomes

$$\boxed{\hat{\boldsymbol{r}} \cdot \left[\boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right] = \nu \nabla_\perp^2 \left\{ \frac{\partial^2 W}{\partial r^2} + \left(2\alpha - \frac{\beta}{3}\right) \frac{\partial W}{\partial r} + \left(\nabla_\perp^2 - \frac{4\alpha\beta}{3} - \frac{4\alpha}{r} - \frac{4}{3}\frac{\partial \beta}{\partial r} - \frac{4\beta}{3r}\right) W \right\}.} \tag{B.67}$$

B.3.3.3 Polar Component

For any tensor $\underline{\underline{\boldsymbol{\mathcal{T}}}}$, the polar component of its divergence is

$$\hat{\boldsymbol{\theta}} \cdot \left(\boldsymbol{\nabla} \cdot \underline{\underline{\boldsymbol{\mathcal{T}}}}\right) = \underbrace{\frac{1}{r^2}\frac{\partial (r^2 \mathcal{T}_{r\theta})}{\partial r}}_{\boxed{1}} + \underbrace{\frac{1}{r\sin\theta}\frac{\partial (\sin\theta\, \mathcal{T}_{\theta\theta})}{\partial \theta}}_{\boxed{2}} + \underbrace{\frac{1}{r\sin\theta}\frac{\partial \mathcal{T}_{\theta\phi}}{\partial \phi}}_{\boxed{3}} + \underbrace{\frac{\mathcal{T}_{r\theta}}{r}}_{\boxed{4}} - \underbrace{\frac{\cos\theta}{r\sin\theta}\mathcal{T}_{\phi\phi}}_{\boxed{5}}. \tag{B.68}$$

In this section, we evaluate (B.68) for $\underline{\underline{\boldsymbol{\mathcal{T}}}} = \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}$ using the definition of $\underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}$ given in equation (B.52) along with the strain rate tensor components of equation (B.53). Starting with the first term, we have

$$\begin{aligned}
\boxed{1} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\, 2\nu\, \hat{\rho} e_{r\theta}\right) = \nu\left(2\alpha + \frac{4}{r} + 2\frac{\partial}{\partial r}\right)(\hat{\rho} e_{r\theta}) \\
&= 2\nu\alpha\, \hat{\rho} e_{r\theta} + \nu\left(\frac{4}{r} + 2\frac{\partial}{\partial r}\right)\left[\frac{1}{2}\frac{\partial (\hat{\rho} u_\theta)}{\partial r} - \frac{\beta}{2}\hat{\rho} u_\theta - \frac{1}{2r}\hat{\rho} u_\theta + \frac{1}{2r}\frac{\partial (\hat{\rho} u_r)}{\partial \theta}\right] \\
&= \underbrace{2\nu\alpha\, \hat{\rho} e_{r\theta}}_{\boxed{1a}} - \underbrace{\nu\left[\beta\left(\frac{\partial}{\partial r} + \frac{2}{r}\right)(\hat{\rho} u_\theta) + \frac{d\beta}{dr}\hat{\rho} u_\theta\right]}_{\boxed{1b}} + \underbrace{\nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)(\hat{\rho} u_\theta)}_{\boxed{1c}} \\
&\quad + \underbrace{\frac{\nu}{r}\left[\left(\frac{\partial^2}{\partial r\, \partial \theta} + \frac{1}{r}\frac{\partial}{\partial \theta}\right)(\hat{\rho} u_r)\right]}_{\boxed{1d}} - \underbrace{\frac{\nu}{r^2}\hat{\rho} u_\theta}_{\boxed{1e}}.
\end{aligned} \tag{B.69}$$

The second term is

$$\boxed{2} = \frac{2\nu}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{1}{r}\frac{\partial(\hat{\rho}u_\theta)}{\partial\theta} + \frac{1}{r}\,\hat{\rho}u_r + \frac{\beta}{3}\,\hat{\rho}u_r\right)\right]$$

$$= \underbrace{\frac{2\nu}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial(\hat{\rho}u_\theta)}{\partial\theta}\right]}_{\boxed{2a}} + \underbrace{\frac{2\nu}{r^2\sin\theta}\frac{\partial(\sin\theta\,\hat{\rho}u_r)}{\partial\theta}}_{\boxed{2b}} + \underbrace{\frac{2\nu\beta}{3r\sin\theta}\frac{\partial(\sin\theta\,\hat{\rho}u_r)}{\partial\theta}}_{\boxed{2c}}, \qquad \text{(B.70)}$$

while the third term is

$$\boxed{3} = \frac{\nu}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{1}{r}\frac{\partial(\hat{\rho}u_\phi)}{\partial\theta} - \frac{\cos\theta}{r\sin\theta}\,\hat{\rho}u_\phi + \frac{1}{r\sin\theta}\frac{\partial(\hat{\rho}u_\theta)}{\partial\phi}\right]$$

$$= \nu\left[\frac{1}{r^2\sin\theta}\frac{\partial^2(\hat{\rho}u_\phi)}{\partial\theta\,\partial\phi} - \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2(\hat{\rho}u_\theta)}{\partial\phi^2}\right]$$

$$= \underbrace{\frac{\nu}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{r\sin\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi}\right]}_{\boxed{3a}} + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2(\hat{\rho}u_\theta)}{\partial\phi^2}}_{\boxed{3b}}. \qquad \text{(B.71)}$$

The last two terms are

$$\boxed{4} = \underbrace{\frac{\nu}{r}\frac{\partial(\hat{\rho}u_\theta)}{\partial r}}_{\boxed{4a}} - \underbrace{\frac{\nu\beta}{r}\,\rho\hat{u}_\theta}_{\boxed{4b}} - \underbrace{\frac{\nu}{r^2}\,\hat{\rho}u_\theta}_{\boxed{4c}} + \underbrace{\frac{\nu}{r^2}\frac{\partial(\hat{\rho}u_r)}{\partial\theta}}_{\boxed{4d}} \qquad \text{(B.72)}$$

and

$$\boxed{5} = -\underbrace{\frac{2\nu\cos\theta}{r^2\sin^2\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi}}_{\boxed{5a}} - \underbrace{\frac{2\nu\cos\theta}{r^2\sin\theta}\,\hat{\rho}u_r}_{\boxed{5b}} - \underbrace{\frac{2\nu\cos^2\theta}{r^2\sin^2\theta}\,\hat{\rho}u_\theta}_{\boxed{5c}} - \underbrace{\frac{2\nu\beta\cos\theta}{3r\sin\theta}\,\hat{\rho}u_r}_{\boxed{5d}}. \qquad \text{(B.73)}$$

To evaluate $\hat{\boldsymbol{\theta}}\cdot\left[\boldsymbol{\nabla}\cdot\underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right] = \boxed{1} + \boxed{2} + \boxed{3} + \boxed{4} + \boxed{5}$ by equation (B.68), we combine pieces from equations (B.69)–(B.73) as indicated in the following expressions:

$$\boxed{1a} = \nu\alpha\left[\frac{\partial(\hat{\rho}u_\theta)}{\partial r} - \beta\,\hat{\rho}u_\theta - \frac{1}{r}\,\hat{\rho}u_\theta + \frac{1}{r}\frac{\partial(\hat{\rho}u_r)}{\partial\theta}\right], \qquad \text{(B.74)}$$

$$\boxed{1b} + \boxed{2c} + \boxed{4b} + \boxed{5d} = \nu\left[\frac{2\beta}{3r}\frac{\partial(\hat{\rho}u_r)}{\partial\theta} - \beta\frac{\partial(\hat{\rho}u_\theta)}{\partial r} - \frac{d\beta}{dr}\,\hat{\rho}u_\theta - \frac{3\beta}{r}\,\hat{\rho}u_\theta\right], \qquad \text{(B.75)}$$

$$\boxed{1c}+\frac{\boxed{2a}}{2}+\boxed{3b}+\boxed{4a}=\nu\nabla^2(\hat{\rho}u_\theta), \tag{B.76}$$

$$\boxed{1d}+\frac{\boxed{2a}}{2}+\boxed{3a}+\boxed{4d}=\frac{\nu}{r}\frac{\partial}{\partial\theta}\underbrace{\left[\boldsymbol{\nabla}\cdot(\hat{\rho}\boldsymbol{u})\right]}_{=0}+\frac{\nu}{r^2\sin^2\theta}\,\hat{\rho}u_\theta, \tag{B.77}$$

$$\boxed{1e}+\boxed{4c}+\boxed{5c}=-\frac{2\nu}{r^2\sin^2\theta}\,\hat{\rho}u_\theta, \tag{B.78}$$

$$\boxed{2b}+\boxed{5b}=\frac{2\nu}{r^2}\frac{\partial(\hat{\rho}u_r)}{\partial\theta}, \tag{B.79}$$

along with

$$\boxed{5a}=-\frac{2\nu\cos\theta}{r^2\sin^2\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi}, \tag{B.80}$$

which is left unchanged. Combining equations (B.74)–(B.80) and regrouping, we obtain

$$\boxed{\begin{aligned}\hat{\boldsymbol{\theta}}\cdot\left[\boldsymbol{\nabla}\cdot\underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right]=\nu\Bigg\{&\nabla^2(\hat{\rho}u_\theta)+\frac{1}{r}\left[\alpha+\frac{2\beta}{3}+\frac{2}{r}\right]\frac{\partial(\hat{\rho}u_r)}{\partial\theta}+(\alpha-\beta)\,\frac{\partial(\hat{\rho}u_\theta)}{\partial r}\\&-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi}-\left[\alpha\beta+\frac{\alpha}{r}+\frac{d\beta}{dr}+\frac{3\beta}{r}+\frac{1}{r^2\sin^2\theta}\right]\,\hat{\rho}u_\theta\Bigg\}\end{aligned}} \tag{B.81}$$

The polar component will be used in combination with the azimuthal component in §B.3.3.5 and §B.3.3.6.

B.3.3.4 Azimuthal Component

For any tensor $\underline{\underline{\boldsymbol{\mathcal{T}}}}$, the azimuthal component of its divergence is

$$\hat{\boldsymbol{\phi}}\cdot\left(\boldsymbol{\nabla}\cdot\underline{\underline{\boldsymbol{\mathcal{T}}}}\right)=\underbrace{\frac{1}{r^2}\frac{\partial(r^2\mathcal{T}_{r\phi})}{\partial r}}_{\boxed{1}}+\underbrace{\frac{1}{r}\frac{\partial\mathcal{T}_{\theta\phi}}{\partial\theta}}_{\boxed{2}}+\underbrace{\frac{1}{r\sin\theta}\frac{\partial\mathcal{T}_{\phi\phi}}{\partial\phi}}_{\boxed{3}}+\underbrace{\frac{\mathcal{T}_{r\phi}}{r}}_{\boxed{4}}+\underbrace{\frac{2\cos\theta}{r\sin\theta}\mathcal{T}_{\theta\phi}}_{\boxed{5}}. \tag{B.82}$$

In this section, we evaluate equation (B.82) for $\underline{\underline{\boldsymbol{\mathcal{T}}}}=\underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}$ using the definition of $\underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}$ given in equation (B.52) along with the strain rate tensor components of equation (B.53).

Starting with the first term, we have

$$\boxed{1} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,2\nu\,\hat{\rho}e_{r\phi}\right) = \nu\left(2\alpha + \frac{4}{r} + 2\frac{\partial}{\partial r}\right)(\hat{\rho}e_{r\phi})$$
$$= 2\nu\alpha\,\hat{\rho}e_{r\phi} + \nu\left(\frac{4}{r} + 2\frac{\partial}{\partial r}\right)\left[\frac{1}{2r\sin\theta}\frac{\partial(\hat{\rho}u_r)}{\partial\phi} + \frac{1}{2}\frac{\partial(\hat{\rho}u_\phi)}{\partial r} - \frac{\beta}{2}\,\hat{\rho}u_\phi - \frac{1}{2r}\,\hat{\rho}u_\phi\right]$$
$$= \underbrace{2\nu\alpha\,\hat{\rho}e_{r\phi}}_{\boxed{1a}} \underbrace{-\,\nu\left[\beta\left(\frac{\partial}{\partial r} + \frac{2}{r}\right)(\hat{\rho}u_\phi) + \frac{d\beta}{dr}\,\hat{\rho}u_\phi\right]}_{\boxed{1b}} + \underbrace{\nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)(\hat{\rho}u_\phi)}_{\boxed{1c}}$$
$$+ \underbrace{\frac{\nu}{r\sin\theta}\left(\frac{\partial^2}{\partial r\,\partial\phi} + \frac{1}{r}\frac{\partial}{\partial\phi}\right)(\hat{\rho}u_r)}_{\boxed{1d}} - \underbrace{\frac{\nu}{r^2}\,\hat{\rho}u_\phi}_{\boxed{1e}}\,. \tag{B.83}$$

The second term is

$$\boxed{2} = \frac{\nu}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{r}\frac{\partial(\hat{\rho}u_\phi)}{\partial\theta} - \frac{\cos\theta}{r\sin\theta}\,\hat{\rho}u_\phi + \frac{1}{r\sin\theta}\frac{\partial(\hat{\rho}u_\theta)}{\partial\phi}\right]$$
$$= \underbrace{\frac{\nu}{r^2}\left(\frac{\partial^2}{\partial\theta^2} - \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right)(\hat{\rho}u_\phi)}_{\boxed{2a}} + \underbrace{\frac{\nu}{r^2\sin^2\theta}\,\hat{\rho}u_\phi}_{\boxed{2b}} + \underbrace{\frac{\nu}{r^2\sin\theta}\left(\frac{\partial^2}{\partial\theta\,\partial\phi} - \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right)(\hat{\rho}u_\theta)}_{\boxed{2c}}, \tag{B.84}$$

while the third term is

$$\boxed{3} = \frac{2\nu}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{1}{r\sin\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi} + \frac{1}{r}\,\hat{\rho}u_r + \frac{\cos\theta}{r\sin\theta}\,\hat{\rho}u_\theta + \frac{\beta}{3}\,\hat{\rho}u_r\right]$$
$$= \underbrace{\frac{2\nu}{r^2\sin^2\theta}\frac{\partial^2(\hat{\rho}u_\phi)}{\partial\phi^2}}_{\boxed{3a}} + \underbrace{\frac{2\nu}{r^2\sin\theta}\frac{\partial(\hat{\rho}u_r)}{\partial\phi}}_{\boxed{3b}} + \underbrace{\frac{2\nu\cos\theta}{r^2\sin^2\theta}\frac{\partial(\hat{\rho}u_\theta)}{\partial\phi}}_{\boxed{3c}} + \underbrace{\frac{2\nu\beta}{3r\sin\theta}\frac{\partial(\hat{\rho}u_r)}{\partial\phi}}_{\boxed{3d}}\,. \tag{B.85}$$

The last two terms are

$$\boxed{4} = \underbrace{\frac{\nu}{r^2\sin\theta}\frac{\partial(\hat{\rho}u_r)}{\partial\phi}}_{\boxed{4a}} + \underbrace{\frac{\nu}{r}\frac{\partial(\hat{\rho}u_\phi)}{\partial r}}_{\boxed{4b}} - \underbrace{\frac{\nu\beta}{r}\,\hat{\rho}u_\phi}_{\boxed{4c}} - \underbrace{\frac{\nu}{r^2}\,\hat{\rho}u_\phi}_{\boxed{4d}} \tag{B.86}$$

and

$$\boxed{5} = \underbrace{\frac{2\nu\cos\theta}{r^2\sin\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\theta}}_{\boxed{5a}} - \underbrace{\frac{2\nu\cos^2\theta}{r^2\sin^2\theta}\,\hat{\rho}u_\phi}_{\boxed{5b}} + \underbrace{\frac{2\nu\cos\theta}{r^2\sin^2\theta}\frac{\partial(\hat{\rho}u_\theta)}{\partial\phi}}_{\boxed{5c}}\,. \tag{B.87}$$

To evaluate $\hat{\boldsymbol{\phi}} \cdot \left[\boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right] = \boxed{1} + \boxed{2} + \boxed{3} + \boxed{4} + \boxed{5}$ by equation (B.82), we combine pieces from equations (B.83)–(B.87) as indicated in the following expressions:

$$\boxed{1a} = \nu\alpha \left[\frac{1}{r\sin\theta}\frac{\partial(\hat{\rho}u_r)}{\partial\phi} + \frac{\partial(\hat{\rho}u_\phi)}{\partial r} - \beta\,\hat{\rho}u_\phi - \frac{1}{r}\,\hat{\rho}u_\phi\right], \tag{B.88}$$

$$\boxed{1b} + \boxed{3d} + \boxed{4c} = \nu\left[\frac{2\beta}{3r\sin\theta}\frac{\partial(\hat{\rho}u_r)}{\partial\phi} - \beta\frac{\partial(\hat{\rho}u_\phi)}{\partial r} - \frac{d\beta}{dr}\,\hat{\rho}u_\phi - \frac{3\beta}{r}\,\hat{\rho}u_\phi\right], \tag{B.89}$$

$$\boxed{1c} + \boxed{2a} + \frac{\boxed{3a}}{2} + \boxed{4b} + \boxed{5a} = \nu\nabla^2(\hat{\rho}u_\phi), \tag{B.90}$$

$$\boxed{1d} + \boxed{2c} + \frac{\boxed{3a}}{2} + \boxed{4a} + \boxed{5c} = \frac{\nu}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\boldsymbol{\nabla}\cdot(\hat{\rho}\boldsymbol{u})\right] = 0, \tag{B.91}$$

$$\begin{aligned}\boxed{1e} + \boxed{2b} + \boxed{4d} + \boxed{5b} &= \frac{\nu}{r^2\sin^2\theta}\left[1 - 2\sin^2\theta - 2\cos^2\theta\right](\hat{\rho}u_\phi) \\ &= -\frac{\nu}{r^2\sin^2\theta}\,\hat{\rho}u_\phi.\end{aligned} \tag{B.92}$$

The remaining terms are unchanged:

$$\boxed{3b} + \boxed{3c} = \nu\left[\frac{2}{r^2\sin\theta}\frac{\partial(\hat{\rho}u_r)}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial(\hat{\rho}u_\theta)}{\partial\phi}\right]. \tag{B.93}$$

Combining equations (B.88)–(B.93) and regrouping, we obtain

$$\boxed{\begin{aligned}\hat{\boldsymbol{\phi}} \cdot \left[\boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right] = \nu\Bigg\{\nabla^2(\hat{\rho}u_\phi) + \frac{1}{r\sin\theta}\left[\alpha + \frac{2\beta}{3} + \frac{2}{r}\right]\frac{\partial(\hat{\rho}u_r)}{\partial\phi} + (\alpha-\beta)\,\frac{\partial(\hat{\rho}u_\phi)}{\partial r} \\ + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial(\hat{\rho}u_\theta)}{\partial\phi} - \left[\alpha\beta + \frac{\alpha}{r} + \frac{d\beta}{dr} + \frac{3\beta}{r} + \frac{1}{r^2\sin^2\theta}\right]\hat{\rho}u_\phi\Bigg\}\end{aligned}} \tag{B.94}$$

B.3.3.5 Perpendicular Divergence

We now use the results of the previous two sections to evaluate $\boldsymbol{\nabla}_\perp \cdot \left[\boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right]$, which serves as the diffusion term in the P equation. By definition,

$$\boldsymbol{\nabla}_\perp \cdot \boldsymbol{A} = \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}, \tag{B.95}$$

such that

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \boldsymbol{\nabla}_\perp \cdot \boldsymbol{A}. \tag{B.96}$$

Thus,

$$\boldsymbol{\nabla}_{\perp}\cdot\left[\boldsymbol{\nabla}\cdot\underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right]=\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\nabla}\cdot\underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right)+\frac{\partial}{\partial\phi}\left(\hat{\boldsymbol{\phi}}\cdot\boldsymbol{\nabla}\cdot\underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right)\right]. \tag{B.97}$$

The polar and azimuthal components of $\boldsymbol{\nabla}\cdot\underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}$ are given by equations (B.81) and (B.94). We rewrite them here and label the individual terms before combining them as per equation (B.97):

$$\hat{\boldsymbol{\theta}}\cdot\left[\boldsymbol{\nabla}\cdot\underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right]=\nu\left\{\underbrace{\nabla^2(\hat{\rho}u_\theta)}_{\boxed{1a}}+\underbrace{\frac{1}{r}\left[\alpha+\frac{2\beta}{3}+\frac{2}{r}\right]\frac{\partial(\hat{\rho}u_r)}{\partial\theta}}_{\boxed{1b}}+\underbrace{(\alpha-\beta)\frac{\partial(\hat{\rho}u_\theta)}{\partial r}}_{\boxed{1c}}\right.$$
$$\left.-\underbrace{\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi}}_{\boxed{1d}}-\underbrace{\left[\alpha\beta+\frac{\alpha}{r}+\frac{d\beta}{dr}+\frac{3\beta}{r}+\frac{1}{r^2\sin^2\theta}\right]\hat{\rho}u_\theta}_{\boxed{1e}}\right\} \tag{B.98}$$

and

$$\hat{\boldsymbol{\phi}}\cdot\left[\boldsymbol{\nabla}\cdot\underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right]=\nu\left\{\underbrace{\nabla^2(\hat{\rho}u_\phi)}_{\boxed{2a}}+\underbrace{\frac{1}{r\sin\theta}\left[\alpha+\frac{2\beta}{3}+\frac{2}{r}\right]\frac{\partial(\hat{\rho}u_r)}{\partial\phi}}_{\boxed{2b}}+\underbrace{(\alpha-\beta)\frac{\partial(\hat{\rho}u_\phi)}{\partial r}}_{\boxed{2c}}\right.$$
$$\left.+\underbrace{\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial(\hat{\rho}u_\theta)}{\partial\phi}}_{\boxed{2d}}-\underbrace{\left[\alpha\beta+\frac{\alpha}{r}+\frac{d\beta}{dr}+\frac{3\beta}{r}+\frac{1}{r^2\sin^2\theta}\right]\hat{\rho}u_\phi}_{\boxed{2e}}\right\}. \tag{B.99}$$

Combining the terms containing $\boxed{\text{1a}}$ and $\boxed{\text{2a}}$ yields:

$$\begin{aligned}
&\frac{\nu}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\,\boxed{\text{1a}}\right]+\frac{\partial\boxed{\text{2a}}}{\partial\phi}\right\}\\
&\quad=\frac{\nu}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\,\nabla^2(\hat{\rho}u_\theta)\right]+\frac{\partial}{\partial\phi}\left[\nabla^2(\hat{\rho}u_\phi)\right]\right\}\\
&\quad=\frac{\nu}{r\sin\theta}\left\{\nabla^2\left[\frac{\partial\,(\sin\theta\,\hat{\rho}u_\theta)}{\partial\theta}+\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi}\right]-\frac{2\cos\theta}{r^2}\frac{\partial^2(\hat{\rho}u_\theta)}{\partial\theta^2}\right.\\
&\qquad\left.+\left(\frac{4\sin\theta}{r^2}-\frac{2}{r^2\sin\theta}\right)\frac{\partial(\hat{\rho}u_\theta)}{\partial\theta}+\frac{2\cos\theta}{r^2}\,\hat{\rho}u_\theta-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial^2(\hat{\rho}u_\theta)}{\partial\phi^2}\right\}\\
&\quad=\frac{\nu}{r\sin\theta}\left\{\sin\theta\,\nabla^2\Big[r\,\boldsymbol{\nabla}_\perp\cdot(\hat{\rho}\boldsymbol{u})\Big]+\frac{2\cos\theta}{r^2\sin\theta}\frac{\partial^2(\hat{\rho}u_\phi)}{\partial\theta\,\partial\phi}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial^2(\hat{\rho}u_\theta)}{\partial\phi^2}\right.\\
&\qquad\left.+\frac{1}{r^2\sin\theta}\frac{\partial(\hat{\rho}u_\theta)}{\partial\theta}-\frac{\cos\theta}{r^2\sin^2\theta}\,\hat{\rho}u_\theta-\frac{1}{r^2\sin^2\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi}\right\}.
\end{aligned}\tag{B.100}$$

Combining the terms containing $\boxed{\text{1b}}$ and $\boxed{\text{2b}}$ yields:

$$\frac{\nu}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\,\boxed{\text{1b}}\right]+\frac{\partial\boxed{\text{2b}}}{\partial\phi}\right\}=\nu\left[\alpha+\frac{2\beta}{3}+\frac{2}{r}\right]\nabla_\perp^2(\hat{\rho}u_r).\tag{B.101}$$

Combining the terms containing $\boxed{\text{1c}}$ and $\boxed{\text{2c}}$ yields:

$$\begin{aligned}
\frac{\nu}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\,\boxed{\text{1c}}\right]+\frac{\partial\boxed{\text{2c}}}{\partial\phi}\right\}&=\frac{\nu(\alpha-\beta)}{r\sin\theta}\frac{\partial}{\partial r}\left[\frac{\partial(\sin\theta\,\hat{\rho}u_\theta)}{\partial\theta}+\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi}\right]\\
&=\frac{\nu(\alpha-\beta)}{r}\frac{\partial}{\partial r}\Big[r\boldsymbol{\nabla}_\perp\cdot(\hat{\rho}\boldsymbol{u})\Big].
\end{aligned}\tag{B.102}$$

Combining the terms containing $\boxed{\text{1d}}$ and $\boxed{\text{2d}}$ yields:

$$\begin{aligned}
&\frac{\nu}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\,\boxed{\text{1d}}\right]+\frac{\partial\boxed{\text{2d}}}{\partial\phi}\right\}\\
&\quad=\frac{\nu}{r\sin\theta}\left[-\frac{\partial}{\partial\theta}\left(\frac{2\cos\theta}{r^2\sin\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi}\right)+\frac{\partial}{\partial\phi}\left(\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial(\hat{\rho}u_\theta)}{\partial\phi}\right)\right]\\
&\quad=\frac{\nu}{r\sin\theta}\left[-\frac{2\cos\theta}{r^2\sin\theta}\frac{\partial^2(\hat{\rho}u_\phi)}{\partial\theta\,\partial\phi}+\frac{2}{r^2\sin^2\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial^2(\hat{\rho}u_\theta)}{\partial\phi^2}\right].
\end{aligned}\tag{B.103}$$

Finally, combining the terms containing $\boxed{1e}$ and $\boxed{2e}$ yields:

$$
\begin{aligned}
&-\frac{\nu}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\,\boxed{1e}\right]+\frac{\partial\,\boxed{2e}}{\partial\phi}\right\} \\
&\quad= -\nu\left[\alpha\beta+\frac{\alpha}{r}+\frac{d\beta}{dr}+\frac{3\beta}{r}\right]\boldsymbol{\nabla}_\perp\cdot(\hat{\rho}\boldsymbol{u}) \\
&\qquad -\frac{\nu}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{\hat{\rho}u_\theta}{r^2\sin\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{\hat{\rho}u_\phi}{r^2\sin^2\theta}\right)\right] \\
&\quad= -\nu\left[\alpha\beta+\frac{\alpha}{r}+\frac{d\beta}{dr}+\frac{3\beta}{r}\right]\boldsymbol{\nabla}_\perp\cdot(\hat{\rho}\boldsymbol{u}) \\
&\qquad +\frac{\nu}{r\sin\theta}\left[-\frac{1}{r^2\sin\theta}\frac{\partial(\hat{\rho}u_\theta)}{\partial\theta}+\frac{\cos\theta}{r^2\sin^2\theta}\,\hat{\rho}u_\theta-\frac{1}{r^2\sin^2\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi}\right].
\end{aligned}
\tag{B.104}
$$

Summing together the terms in equations (B.100)–(B.104) yields

$$
\begin{aligned}
\boldsymbol{\nabla}_\perp\cdot\left[\boldsymbol{\nabla}\cdot\underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right] = \nu\Bigg\{&\underbrace{\frac{1}{r}\nabla^2\left[r\,\boldsymbol{\nabla}_\perp\cdot(\hat{\rho}\boldsymbol{u})\right]}_{\boxed{3a}}-\underbrace{\left[\alpha\beta+\frac{\alpha}{r}+\frac{d\beta}{dr}+\frac{3\beta}{r}\right]\boldsymbol{\nabla}_\perp\cdot(\hat{\rho}\boldsymbol{u})}_{\boxed{3b}} \\
&+\underbrace{\frac{\alpha-\beta}{r}\frac{\partial}{\partial r}\left[r\boldsymbol{\nabla}_\perp\cdot(\hat{\rho}\boldsymbol{u})\right]}_{\boxed{3c}}+\underbrace{\left[\alpha+\frac{2\beta}{3}+\frac{2}{r}\right]\nabla_\perp^2(\hat{\rho}u_r)}_{\boxed{3d}}\Bigg\}.
\end{aligned}
\tag{B.105}
$$

Using equation (B.96) and the fact that $\boldsymbol{\nabla}\cdot(\hat{\rho}\boldsymbol{u})=0$, we can make the substitution

$$
\boldsymbol{\nabla}_\perp\cdot(\hat{\rho}\boldsymbol{u}) = -\frac{1}{r^2}\frac{\partial(r^2\,\hat{\rho}u_r)}{\partial r} = -\frac{\partial(\hat{\rho}u_r)}{\partial r}-\frac{2}{r}\,\hat{\rho}u_r
\tag{B.106}
$$

such that only $\hat{\rho}u_r$ remains, and then use equation (B.45) to eliminate $\hat{\rho}u_r$ in favor of W. Once in terms of W, the commutation relations (B.49)–(B.51) are used to interchange the $\nabla_\perp^2$ operator with r-derivatives. We now examine each term in equation (B.105) above, starting with the first term:

$$
\begin{aligned}
\boxed{3a} &= \frac{1}{r}\nabla^2\left[r\,\boldsymbol{\nabla}_\perp\cdot(\hat{\rho}\boldsymbol{u})\right] \\
&= -\frac{1}{r}\nabla^2\left[r\,\frac{\partial(\hat{\rho}u_r)}{\partial r}+2\,\hat{\rho}u_r\right] \\
&= -\frac{1}{r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\nabla_\perp^2\right]\left[r\,\frac{\partial(\hat{\rho}u_r)}{\partial r}+2\,\hat{\rho}u_r\right] \\
&= -\left[\frac{\partial^3(\hat{\rho}u_r)}{\partial r^3}+\frac{6}{r}\frac{\partial^2(\hat{\rho}u_r)}{\partial r^2}+\frac{6}{r^2}\frac{\partial(\hat{\rho}u_r)}{\partial r}+\nabla_\perp^2\frac{\partial(\hat{\rho}u_r)}{\partial r}+\frac{2}{r}\nabla_\perp^2(\hat{\rho}u_r)\right] \\
&= \nabla_\perp^2\left[\frac{\partial^3 W}{\partial r^3}+\nabla_\perp^2\frac{\partial W}{\partial r}\right].
\end{aligned}
\tag{B.107}
$$

The next two terms are

$$\boxed{3\text{b}} = -\left[\alpha\beta + \frac{\alpha}{r} + \frac{d\beta}{dr} + \frac{3\beta}{r}\right] \boldsymbol{\nabla}_\perp \cdot (\hat{\rho}\boldsymbol{u})$$
$$= \left[\alpha\beta + \frac{\alpha}{r} + \frac{d\beta}{dr} + \frac{3\beta}{r}\right] \left[\frac{\partial(\hat{\rho}u_r)}{\partial r} + \frac{2}{r}\,\hat{\rho}u_r\right]$$
$$= -\nabla_\perp^2 \left[\alpha\beta + \frac{\alpha}{r} + \frac{d\beta}{dr} + \frac{3\beta}{r}\right] \frac{\partial W}{\partial r} \tag{B.108}$$

and

$$\boxed{3\text{c}} = \frac{\alpha - \beta}{r}\frac{\partial}{\partial r}\left[r\boldsymbol{\nabla}_\perp \cdot (\hat{\rho}\boldsymbol{u})\right]$$
$$= -(\alpha - \beta)\left[\frac{\partial^2(\hat{\rho}u_r)}{\partial r^2} + \frac{3}{r}\frac{\partial(\hat{\rho}u_r)}{\partial r}\right]$$
$$= \nabla_\perp^2 \left[(\alpha - \beta)\left(\frac{\partial^2 W}{\partial r^2} - \frac{1}{r}\frac{\partial W}{\partial r}\right)\right]. \tag{B.109}$$

Lastly, the fourth term is simply

$$\boxed{3\text{d}} = \left[\alpha + \frac{2\beta}{3} + \frac{2}{r}\right] \nabla_\perp^2(\hat{\rho}u_r) = -\nabla_\perp^2 \left[\alpha + \frac{2\beta}{3} + \frac{2}{r}\right] \nabla_\perp^2 W. \tag{B.110}$$

Combining equations (B.107)–(B.110) and regrouping, we obtain

$$\boxed{\begin{aligned}\boldsymbol{\nabla}_\perp \cdot \left[\boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right] = \nu\nabla_\perp^2 \Bigg\{ &\frac{\partial^3 W}{\partial r^3} + (\alpha - \beta)\frac{\partial^2 W}{\partial r^2} \\ &- \left[\alpha\beta + \frac{2\alpha}{r} + \frac{d\beta}{dr} + \frac{2\beta}{r} - \nabla_\perp^2\right] \frac{\partial W}{\partial r} \\ &- \left[\alpha + \frac{2\beta}{3} + \frac{2}{r}\right] \nabla_\perp^2 W \Bigg\}.\end{aligned}} \tag{B.111}$$

B.3.3.6 Radial Component of Curl

We now use the results of §B.3.3.3 and §B.3.3.4 to evaluate $\hat{\boldsymbol{r}} \cdot \boldsymbol{\nabla} \times \left[\boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right]$, which serves as the diffusion term in the Z equation. By definition,

$$\hat{\boldsymbol{r}} \cdot \boldsymbol{\nabla} \times \left[\boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right] = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,\hat{\boldsymbol{\phi}} \cdot \boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right) - \frac{\partial}{\partial\phi}\left(\hat{\boldsymbol{\theta}} \cdot \boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right)\right]. \tag{B.112}$$

The polar and azimuthal components of $\boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}$ are given by equations (B.81) and (B.94). We rewrite them here and label the individual terms before combining them as per equation (B.112):

$$\hat{\boldsymbol{\theta}} \cdot \left[\boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right] = \nu \Bigg\{ \underbrace{\nabla^2(\hat{\rho} u_\theta)}_{\boxed{\text{1a}}} + \underbrace{\frac{1}{r}\left[\alpha + \frac{2\beta}{3} + \frac{2}{r}\right] \frac{\partial(\hat{\rho} u_r)}{\partial \theta}}_{\boxed{\text{1b}}} + \underbrace{(\alpha - \beta)\frac{\partial(\hat{\rho} u_\theta)}{\partial r}}_{\boxed{\text{1c}}}$$
$$- \underbrace{\frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial(\hat{\rho} u_\phi)}{\partial \phi}}_{\boxed{\text{1d}}} - \underbrace{\left[\alpha\beta + \frac{\alpha}{r} + \frac{d\beta}{dr} + \frac{3\beta}{r} + \frac{1}{r^2\sin^2\theta}\right] \hat{\rho} u_\theta}_{\boxed{\text{1e}}} \Bigg\} \tag{B.113}$$

and

$$\hat{\boldsymbol{\phi}} \cdot \left[\boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right] = \nu \Bigg\{ \underbrace{\nabla^2(\hat{\rho} u_\phi)}_{\boxed{\text{2a}}} + \underbrace{\frac{1}{r\sin\theta}\left[\alpha + \frac{2\beta}{3} + \frac{2}{r}\right] \frac{\partial(\hat{\rho} u_r)}{\partial \phi}}_{\boxed{\text{2b}}} + \underbrace{(\alpha - \beta)\frac{\partial(\hat{\rho} u_\phi)}{\partial r}}_{\boxed{\text{2c}}}$$
$$+ \underbrace{\frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial(\hat{\rho} u_\theta)}{\partial \phi}}_{\boxed{\text{2d}}} - \underbrace{\left[\alpha\beta + \frac{\alpha}{r} + \frac{d\beta}{dr} + \frac{3\beta}{r} + \frac{1}{r^2\sin^2\theta}\right] \hat{\rho} u_\phi}_{\boxed{\text{2e}}} \Bigg\}. \tag{B.114}$$

Combining the terms containing $\boxed{\text{1a}}$ and $\boxed{\text{2a}}$ yields:

$$\begin{aligned}
&\frac{\nu}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\,\boxed{\text{2a}}\right] - \frac{\partial\,\boxed{\text{1a}}}{\partial\phi}\right\} \\
&\quad= \frac{\nu}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\,\nabla^2(\hat{\rho} u_\phi)\right] - \frac{\partial}{\partial\phi}\left[\nabla^2(\hat{\rho} u_\theta)\right]\right\} \\
&\quad= \frac{\nu}{r\sin\theta}\Bigg\{\nabla^2\left[\frac{\partial\left(\sin\theta\,\hat{\rho} u_\phi\right)}{\partial\theta} - \frac{\partial(\hat{\rho} u_\theta)}{\partial\phi}\right] - \frac{2\cos\theta}{r^2}\frac{\partial^2(\hat{\rho} u_\phi)}{\partial\theta^2} \\
&\qquad + \left(\frac{4\sin\theta}{r^2} - \frac{2}{r^2\sin\theta}\right)\frac{\partial(\hat{\rho} u_\phi)}{\partial\theta} + \frac{2\cos\theta}{r^2}\hat{\rho} u_\phi - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial^2(\hat{\rho} u_\phi)}{\partial\phi^2}\Bigg\} \\
&\quad= \frac{\nu}{r\sin\theta}\Bigg\{\sin\theta\,\nabla^2\Big[r\,\hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla}\times\hat{\rho}\boldsymbol{u}\Big] - \frac{2\cos\theta}{r^2\sin\theta}\frac{\partial^2(\hat{\rho} u_\theta)}{\partial\theta\,\partial\phi} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial^2(\hat{\rho} u_\phi)}{\partial\phi^2} \\
&\qquad + \frac{1}{r^2\sin\theta}\frac{\partial(\hat{\rho} u_\phi)}{\partial\theta} - \frac{\cos\theta}{r^2\sin^2\theta}\hat{\rho} u_\phi + \frac{1}{r^2\sin^2\theta}\frac{\partial(\hat{\rho} u_\theta)}{\partial\phi}\Bigg\}.
\end{aligned} \tag{B.115}$$

Combining the terms containing $\boxed{\text{1b}}$ and $\boxed{\text{2b}}$ yields:

$$\frac{\nu}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\,\boxed{\text{2b}}\right] - \frac{\partial\,\boxed{\text{1b}}}{\partial\phi}\right\} = 0. \tag{B.116}$$

Combining the terms containing $\boxed{\text{1c}}$ and $\boxed{\text{2c}}$ yields:

$$\frac{\nu}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\,\boxed{\text{2c}}\right]-\frac{\partial\,\boxed{\text{1c}}}{\partial\phi}\right\}=\frac{\nu(\alpha-\beta)}{r\sin\theta}\frac{\partial}{\partial r}\left[\frac{\partial(\sin\theta\,\hat{\rho}u_\phi)}{\partial\theta}-\frac{\partial(\hat{\rho}u_\theta)}{\partial\phi}\right]$$
$$=\frac{\nu(\alpha-\beta)}{r}\frac{\partial}{\partial r}\Big[r\,\hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla}\times\hat{\rho}\boldsymbol{u}\Big]. \tag{B.117}$$

Combining the terms containing $\boxed{\text{1d}}$ and $\boxed{\text{2d}}$ yields:

$$\frac{\nu}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\,\boxed{\text{2d}}\right]-\frac{\partial\,\boxed{\text{1d}}}{\partial\phi}\right\}$$
$$=\frac{\nu}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{2\cos\theta}{r^2\sin\theta}\frac{\partial(\hat{\rho}u_\theta)}{\partial\phi}\right)+\frac{\partial}{\partial\phi}\left(\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\phi}\right)\right]$$
$$=\frac{\nu}{r\sin\theta}\left[\frac{2\cos\theta}{r^2\sin\theta}\frac{\partial^2(\hat{\rho}u_\theta)}{\partial\theta\,\partial\phi}-\frac{2}{r^2\sin^2\theta}\frac{\partial(\hat{\rho}u_\theta)}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial^2(\hat{\rho}u_\phi)}{\partial\phi^2}\right] \tag{B.118}$$

Finally, combining the terms containing $\boxed{\text{1e}}$ and $\boxed{\text{2e}}$ yields:

$$\frac{\nu}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\,\boxed{\text{2e}}\right]-\frac{\partial\,\boxed{\text{1e}}}{\partial\phi}\right\}$$
$$=-\nu\left[\alpha\beta+\frac{\alpha}{r}+\frac{d\beta}{dr}+\frac{3\beta}{r}\right]\hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla}\times\hat{\rho}\boldsymbol{u}$$
$$-\frac{\nu}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{\hat{\rho}u_\phi}{r^2\sin\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{\hat{\rho}u_\theta}{r^2\sin^2\theta}\right)\right]$$
$$=-\nu\left[\alpha\beta+\frac{\alpha}{r}+\frac{d\beta}{dr}+\frac{3\beta}{r}\right]\hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla}\times\hat{\rho}\boldsymbol{u}$$
$$+\frac{\nu}{r\sin\theta}\left[\frac{\cos\theta}{r^2\sin^2\theta}\hat{\rho}u_\phi+\frac{1}{r^2\sin^2\theta}\frac{\partial(\hat{\rho}u_\theta)}{\partial\phi}-\frac{1}{r^2\sin\theta}\frac{\partial(\hat{\rho}u_\phi)}{\partial\theta}\right]. \tag{B.119}$$

Summing together the terms in equations (B.115)–(B.119) yields

$$\hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla}\times\left[\boldsymbol{\nabla}\cdot\hat{\underline{\underline{\boldsymbol{\mathcal{D}}}}}\right]=\nu\left\{\frac{1}{r}\nabla^2\Big[r\,\hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla}\times\hat{\rho}\boldsymbol{u}\Big]-\left[\alpha\beta+\frac{\alpha}{r}+\frac{d\beta}{dr}+\frac{3\beta}{r}\right]\hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla}\times\hat{\rho}\boldsymbol{u}\right.$$
$$\left.+\frac{\alpha-\beta}{r}\frac{\partial}{\partial r}\Big[r\,\hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla}\times\hat{\rho}\boldsymbol{u}\Big]\right\}. \tag{B.120}$$

We now eliminate $\hat{\rho}\boldsymbol{u}$ in favor of Z by substituting $\hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla}\times\hat{\rho}\boldsymbol{u}=-\nabla_\perp^2 Z$ by equation (B.46) to obtain

$$\begin{aligned}
\hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla}\times\Big[\boldsymbol{\nabla}\cdot\underline{\underline{\hat{\mathcal{D}}}}\Big] &= -\nu\left\{\frac{1}{r}\nabla^2\Big[r\nabla_\perp^2 Z\Big]-\left[\alpha\beta+\frac{\alpha}{r}+\frac{d\beta}{dr}+\frac{3\beta}{r}\right]\nabla_\perp^2 Z\right.\\
&\qquad\left.+\frac{\alpha-\beta}{r}\frac{\partial}{\partial r}\Big[r\nabla_\perp^2 Z\Big]\right\}\\
&= -\nu\left\{\frac{1}{r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\nabla_\perp^2\right]\Big[r\nabla_\perp^2 Z\Big]\right.\\
&\qquad\left.-\nabla_\perp^2\left[\alpha\beta+\frac{\alpha}{r}+\frac{d\beta}{dr}+\frac{3\beta}{r}\right]Z+\frac{\alpha-\beta}{r}\frac{\partial}{\partial r}\Big[r\nabla_\perp^2 Z\Big]\right\}\\
&= -\nu\left\{\left[\frac{\partial^2}{\partial r^2}+\frac{4}{r}\frac{\partial}{\partial r}+\frac{2}{r^2}+\nabla_\perp^2\right]\nabla_\perp^2 Z\right.\\
&\qquad\left.-\nabla_\perp^2\left[\alpha\beta+\frac{\alpha}{r}+\frac{d\beta}{dr}+\frac{3\beta}{r}\right]Z+(\alpha-\beta)\left[\frac{\partial}{\partial r}+\frac{1}{r}\right]\nabla_\perp^2 Z.\right\}
\end{aligned}$$

Using the commutation identities (B.49) and (B.50), we finally obtain

$$\boxed{\begin{aligned}\hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla}\times\Big[\boldsymbol{\nabla}\cdot\underline{\underline{\hat{\mathcal{D}}}}\Big] = -\nu\nabla_\perp^2\left\{\frac{\partial^2 Z}{\partial r^2}+(\alpha-\beta)\frac{\partial Z}{\partial r}\right.\\
\left.-\left[\alpha\beta+\frac{2\alpha}{r}+\frac{d\beta}{dr}+\frac{2\beta}{r}+\nabla_\perp^2\right]Z\right\}.\end{aligned}} \tag{B.121}$$

B.3.4 Derivation of the W Equation

The evolution equation for W is obtained by taking $\hat{\boldsymbol{r}}\cdot$ each term in the anelastic momentum equation (B.8). Applying $\hat{\boldsymbol{r}}\cdot$ to the time-derivative term gives

$$\hat{\rho}\,\hat{\boldsymbol{r}}\cdot\left(\frac{\partial\boldsymbol{u}}{\partial t}\right)=\frac{\partial}{\partial t}(\hat{\boldsymbol{r}}\cdot\hat{\rho}\boldsymbol{u})=-\frac{\partial}{\partial t}(\nabla_\perp^2 W)=-\nabla_\perp^2\frac{\partial W}{\partial t}, \tag{B.122}$$

where we have used equation (B.45). As a result, a schematic representation of the W equation is

$$\boxed{-\nabla_\perp^2\frac{\partial W}{\partial t}=W_{\mathrm{PG}}+W_{\mathrm{GRAV}}+W_{\mathrm{DIFF}}+W_{\mathrm{COR}}+W_{\mathrm{ADV}}.} \tag{B.123}$$

We now compute the five terms on the right-hand side.

The pressure gradient and gravitational force terms are simply

$$W_{\mathrm{PG}}=\hat{\boldsymbol{r}}\cdot\Big(-\boldsymbol{\nabla}p\Big)=-\frac{\partial p}{\partial r}, \tag{B.124}$$

and

$$W_{\text{GRAV}} = \hat{\boldsymbol{r}} \cdot \left(-\rho g \, \hat{\boldsymbol{r}} \right) = -\rho g. \tag{B.125}$$

Equation (B.67) gives the diffusion term in terms of W:

$$\begin{aligned} W_{\text{DIFF}} &= \hat{\boldsymbol{r}} \cdot \left[\boldsymbol{\nabla} \cdot \underline{\underline{\hat{\mathcal{D}}}} \right] \\ &= \nu_{\text{eff}} \nabla_\perp^2 \left\{ \frac{\partial^2 W}{\partial r^2} + \left(2\alpha - \frac{\beta}{3} \right) \frac{\partial W}{\partial r} + \left(\nabla_\perp^2 - \frac{4\alpha\beta}{3} - \frac{4\alpha}{r} - \frac{4}{3} \frac{\partial \beta}{\partial r} - \frac{4\beta}{3r} \right) W \right\}, \end{aligned} \tag{B.126}$$

where $\alpha = \dfrac{d \ln \nu_{\text{eff}}}{dr}$ and $\beta = \dfrac{d \ln \hat{\rho}}{dr}$. The Coriolis term can also be expanded in terms of the streamfunctions:

$$\begin{aligned} W_{\text{COR}} &= 2\hat{\rho} \, \hat{\boldsymbol{r}} \cdot (\boldsymbol{u} \times \boldsymbol{\Omega}) \\ &= 2\Omega \sin\theta \, \hat{\rho} u_\phi \\ &= \frac{2\Omega}{r} \left[\frac{\partial^2 W}{\partial r \partial \phi} - \sin\theta \frac{\partial Z}{\partial \theta} \right]. \qquad \text{by equation (B.45)} \end{aligned} \tag{B.127}$$

Finally, the advection term can be expanded in terms of the three components of $\boldsymbol{u}$:

$$\begin{aligned} W_{\text{ADV}} &= -\hat{\rho} \, \hat{\boldsymbol{r}} \cdot \left[(\boldsymbol{u} \cdot \boldsymbol{\nabla}) \boldsymbol{u} \right] \\ &= -\hat{\rho} \left[(\boldsymbol{u} \cdot \boldsymbol{\nabla}) u_r - \frac{u_\theta^2}{r} - \frac{u_\phi^2}{r} \right] \\ &= -\hat{\rho} \left[\left(u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + \frac{u_\phi}{r \sin\theta} \frac{\partial}{\partial \phi} \right) u_r - \frac{u_\theta^2}{r} - \frac{u_\phi^2}{r} \right]. \end{aligned} \tag{B.128}$$

B.3.5 Derivation of the $\boldsymbol{P}$ Equation

The evolution equation for P is obtained by taking $\boldsymbol{\nabla}_\perp \cdot$ each term in the anelastic momentum equation (B.8), where the expression $\boldsymbol{\nabla}_\perp \cdot \boldsymbol{A}$ equals

$$\boldsymbol{\nabla}_\perp \cdot \boldsymbol{A} = \frac{1}{r \sin\theta} \frac{\partial (\sin\theta \, A_\theta)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi}, \tag{B.129}$$

such that

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{1}{r^2} \frac{\partial (r^2 \, A_r)}{\partial r} + \boldsymbol{\nabla}_\perp \cdot \boldsymbol{A}. \tag{B.130}$$

The time-derivative term simplifies to

$$\begin{aligned}
\boldsymbol{\nabla}_\perp \cdot \left(\hat{\rho}\frac{\partial \boldsymbol{u}}{\partial t}\right) &= \frac{\partial}{\partial t}\Big[\boldsymbol{\nabla}_\perp \cdot (\hat{\rho}\boldsymbol{u})\Big] \\
&= \frac{\partial}{\partial t}\left[\boldsymbol{\nabla} \cdot (\hat{\rho}\boldsymbol{u}) - \frac{1}{r^2}\frac{\partial (r^2 \hat{\rho} u_r)}{\partial r}\right] && \text{by equation (B.130)} \\
&= -\frac{\partial}{\partial t}\left[\frac{\partial (\hat{\rho} u_r)}{\partial r} + \frac{2\hat{\rho} u_r}{r}\right] && \text{by equation (B.4)} \\
&= \frac{\partial}{\partial t}\left[\frac{\partial (\nabla_\perp^2 W)}{\partial r} + \frac{2}{r}\nabla_\perp^2 W\right] && \text{by equation (B.45)} \\
&= \nabla_\perp^2 \frac{\partial}{\partial t}\left(\frac{\partial W}{\partial r}\right). && \text{by equation (B.49)} \qquad \text{(B.131)}
\end{aligned}$$

Therefore, the P equation is represented by

$$\boxed{\nabla_\perp^2 \frac{\partial}{\partial t}\left(\frac{\partial W}{\partial r}\right) = P_{\text{PG}} + P_{\text{DIFF}} + P_{\text{COR}} + P_{\text{ADV}}.} \qquad \text{(B.132)}$$

Note that there is no gravity term since the quantity $\boldsymbol{\nabla}_\perp \cdot (\rho g\, \hat{\boldsymbol{r}})$ vanishes. We now list the remaining terms on the right-hand side in order.

The pressure gradient term simplifies to

$$P_{\text{PG}} = -\boldsymbol{\nabla}_\perp \cdot (\boldsymbol{\nabla} p) = -\left[\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} p) - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial p}{\partial r}\right)\right] = -\nabla_\perp^2 p, \qquad \text{(B.133)}$$

by equations (B.47) and (B.130). The diffusion term is given by equation (B.111):

$$\begin{aligned}
P_{\text{DIFF}} &= \boldsymbol{\nabla}_\perp \cdot \left[\boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right] \\
&= \nu_{\text{eff}} \nabla_\perp^2 \left\{\frac{\partial^3 W}{\partial r^3} + (\alpha - \beta)\frac{\partial^2 W}{\partial r^2} - \left[\alpha\beta + \frac{2\alpha}{r} + \frac{d\beta}{dr} + \frac{2\beta}{r} - \nabla_\perp^2\right]\frac{\partial W}{\partial r}\right. \\
&\qquad \left. - \left[\alpha + \frac{2\beta}{3} + \frac{2}{r}\right]\nabla_\perp^2 W\right\}. \qquad \text{(B.134)}
\end{aligned}$$

The Coriolis term can be simplified by using

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times Z\, \hat{\boldsymbol{r}} &= \boldsymbol{\nabla} \times \left(\frac{1}{r\sin\theta}\frac{\partial Z}{\partial \phi}\,\hat{\boldsymbol{\theta}} - \frac{1}{r}\frac{\partial Z}{\partial \theta}\,\hat{\boldsymbol{\phi}}\right) \\
&= -\left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial Z}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(\frac{1}{r\sin\theta}\frac{\partial Z}{\partial \phi}\right)\right]\hat{\boldsymbol{r}} \\
&\quad + \frac{1}{r}\frac{\partial^2 Z}{\partial r \partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial^2 Z}{\partial r \partial \phi}\,\hat{\boldsymbol{\phi}} \\
&= -(\nabla_\perp^2 Z)\,\hat{\boldsymbol{r}} + \boldsymbol{\nabla}\left(\frac{\partial Z}{\partial r}\right) - \frac{\partial^2 Z}{\partial r^2}\,\hat{\boldsymbol{r}} \qquad \text{(B.135)}
\end{aligned}$$

and

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times W\,\hat{\boldsymbol{r}} = \boldsymbol{\nabla} \times \left[-(\nabla_\perp^2 W)\,\hat{\boldsymbol{r}} + \boldsymbol{\nabla}\left(\frac{\partial W}{\partial r}\right) - \frac{\partial^2 W}{\partial r^2}\,\hat{\boldsymbol{r}}\right]$$
$$= -\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\nabla_\perp^2 W + \frac{\partial^2 W}{\partial r^2}\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\frac{\partial}{\partial\theta}\left[\nabla_\perp^2 W + \frac{\partial^2 W}{\partial r^2}\right]\hat{\boldsymbol{\phi}} \tag{B.136}$$

so that

$$\begin{aligned}
P_{\mathrm{COR}} &= \boldsymbol{\nabla}_\perp \cdot \left[2\hat{\rho}\boldsymbol{u} \times \boldsymbol{\Omega}\right] \\
&= 2\boldsymbol{\Omega} \cdot \left[\boldsymbol{\nabla} \times (\hat{\rho}\boldsymbol{u})\right] - \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\left[\hat{\boldsymbol{r}} \cdot (2\hat{\rho}\boldsymbol{u} \times \boldsymbol{\Omega})\right] \\
&= 2\Omega(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) \cdot \left[\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times W\,\hat{\boldsymbol{r}} + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times Z\,\hat{\boldsymbol{r}}\right] \\
&\quad - \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\left[2\Omega\sin\theta\,\hat{\rho}u_\phi\right] \\
&= 2\Omega\left[\frac{1}{r}\left(\nabla_\perp^2 + \frac{\partial^2}{\partial r^2}\right)\frac{\partial W}{\partial\phi} - \cos\theta\,\nabla_\perp^2 Z - \frac{\sin\theta}{r}\frac{\partial^2 Z}{\partial r\partial\theta}\right] \\
&\quad - \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\left[\frac{2\Omega}{r}\left(\frac{\partial^2 W}{\partial r\partial\phi} - \sin\theta\frac{\partial^2 W}{\partial r\partial\theta}\right)\right].
\end{aligned} \tag{B.137}$$

The advection term can be expanded in terms of the three components of $\boldsymbol{u}$:

$$\begin{aligned}
P_{\mathrm{ADV}} &= -\boldsymbol{\nabla}_\perp \cdot \left[\hat{\rho}(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}\right] \\
&= -\frac{\hat{\rho}}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\,\hat{\boldsymbol{\theta}} \cdot \left[(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}\right]\right] + \frac{\partial}{\partial\phi}\left[\hat{\boldsymbol{\phi}} \cdot \left[(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}\right]\right]\right\} \\
&= -\frac{\hat{\rho}}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\left((\boldsymbol{u} \cdot \boldsymbol{\nabla})u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\phi^2\cos\theta}{r\sin\theta}\right)\right]\right. \\
&\qquad \left. + \frac{\partial}{\partial\phi}\left[(\boldsymbol{u} \cdot \boldsymbol{\nabla})u_\phi + \frac{u_r u_\phi}{r} + \frac{u_\theta u_\phi\cos\theta}{r\sin\theta}\right]\right\} \\
&= -\frac{\hat{\rho}}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\left(u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{u_r u_\theta}{r} - \frac{u_\phi^2\cos\theta}{r\sin\theta}\right)\right]\right. \\
&\qquad \left. + \frac{\partial}{\partial\phi}\left[u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r u_\phi}{r} + \frac{u_\theta u_\phi\cos\theta}{r\sin\theta}\right]\right\}.
\end{aligned} \tag{B.138}$$

B.3.6 Derivation of the Z Equation

The evolution equation for Z is obtained by taking $\hat{\boldsymbol{r}} \cdot \boldsymbol{\nabla}\times$ each term in the anelastic momentum equation (B.8). Applying $\hat{\boldsymbol{r}} \cdot \boldsymbol{\nabla}\times$ to the time-derivative term gives

$$\hat{\boldsymbol{r}} \cdot \boldsymbol{\nabla} \times \frac{\partial(\hat{\rho}\boldsymbol{u})}{\partial t} = \frac{\partial}{\partial t}(\, \hat{\boldsymbol{r}} \cdot \boldsymbol{\nabla} \times \hat{\rho}\boldsymbol{u}) = -\frac{\partial}{\partial t}(\nabla_\perp^2 Z) = -\nabla_\perp^2 \frac{\partial Z}{\partial t}, \tag{B.139}$$

where we have used equation (B.46). As a result, a schematic representation of the Z equation is

$$\boxed{-\nabla_\perp^2 \frac{\partial Z}{\partial t} = Z_{\mathrm{DIFF}} + Z_{\mathrm{COR}} + Z_{\mathrm{ADV}}.} \tag{B.140}$$

Note that there is no pressure gradient term since $\boldsymbol{\nabla} \times \boldsymbol{\nabla} p = 0$, and that the gravity term vanishes since $\boldsymbol{\nabla} \times (\rho g \, \hat{\boldsymbol{r}})$ has no $\hat{\boldsymbol{r}}$-component. We now compute the three terms on the right-hand side in order.

The diffusion term is given by equation (B.121):

$$\begin{aligned} Z_{\mathrm{DIFF}} &= \hat{\boldsymbol{r}} \cdot \boldsymbol{\nabla} \times \left[\boldsymbol{\nabla} \cdot \underline{\underline{\hat{\boldsymbol{\mathcal{D}}}}}\right] \\ &= -\nu_{\mathrm{eff}} \nabla_\perp^2 \left\{ \frac{\partial^2 Z}{\partial r^2} + (\alpha - \beta)\frac{\partial Z}{\partial r} - \left[\alpha\beta + \frac{2\alpha}{r} + \frac{d\beta}{dr} + \frac{2\beta}{r} + \nabla_\perp^2\right] Z \right\}. \end{aligned} \tag{B.141}$$

The Coriolis term simplifies to

$$\begin{aligned} Z_{\mathrm{COR}} &= \hat{\boldsymbol{r}} \cdot \boldsymbol{\nabla} \times \left[(2\hat{\rho}\boldsymbol{u}) \times \boldsymbol{\Omega}\right] \\ &= 2\,\hat{\boldsymbol{r}} \cdot \left[(\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})(\hat{\rho}\boldsymbol{u})\right] \\ &= 2\Omega \left[\cos\theta \frac{\partial(\hat{\rho} u_r)}{\partial r} - \frac{\sin\theta}{r}\frac{\partial(\hat{\rho} u_r)}{\partial \theta} + \frac{\hat{\rho} u_\theta \sin\theta}{r}\right] \\ &= 2\Omega \left[-\cos\theta \frac{\partial}{\partial r}(\nabla_\perp^2 W) + \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}(\nabla_\perp^2 W) + \frac{\sin\theta}{r^2}\frac{\partial^2 W}{\partial r\,\partial \theta} + \frac{1}{r^2}\frac{\partial Z}{\partial \phi}\right]. \end{aligned} \tag{B.142}$$

The advection term can be expanded in terms of the three components of $\boldsymbol{u}$:

$$\begin{aligned}
Z_{\mathrm{ADV}} &= -\,\hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla}\times\Big[\hat{\rho}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\Big] \\
&= -\frac{\hat{\rho}}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\Big[\sin\theta\;\hat{\boldsymbol{\phi}}\cdot\Big[(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\Big]\Big]-\frac{\partial}{\partial\phi}\Big[\hat{\boldsymbol{\theta}}\cdot\Big[(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\Big]\Big]\right\} \\
&= -\frac{\hat{\rho}}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\left((\boldsymbol{u}\cdot\boldsymbol{\nabla})u_\phi+\frac{u_r u_\phi}{r}+\frac{u_\theta u_\phi\cos\theta}{r\sin\theta}\right)\right]\right. \\
&\qquad\left.-\frac{\partial}{\partial\phi}\left[(\boldsymbol{u}\cdot\boldsymbol{\nabla})u_\theta+\frac{u_r u_\theta}{r}-\frac{u_\phi^2\cos\theta}{r\sin\theta}\right]\right\} \\
&= -\frac{\hat{\rho}}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\sin\theta\left(u_r\frac{\partial u_\phi}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\phi}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\frac{u_r u_\phi}{r}+\frac{u_\theta u_\phi\cos\theta}{r\sin\theta}\right)\right]\right. \\
&\qquad\left.-\frac{\partial}{\partial\phi}\left[u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}+\frac{u_r u_\theta}{r}-\frac{u_\phi^2\cos\theta}{r\sin\theta}\right]\right\} \qquad \text{(B.143)}
\end{aligned}$$

B.3.7 Derivation of the S Equation

The evolution equation for S is simply equation (B.12),

$$\boxed{\hat{\rho}\hat{T}\frac{\partial s}{\partial t} = S_{\mathrm{FLUX}} + S_{\mathrm{ADV}} + S_{\mathrm{DIFF}}.} \qquad \text{(B.144)}$$

where the entropy and radiative fluxes are given by

$$\begin{aligned}
S_{\mathrm{FLUX}} &= -\boldsymbol{\nabla}\cdot\hat{\boldsymbol{q}}_{\mathrm{eff}} \\
&= -\kappa_r\hat{\rho}c_p\boldsymbol{\nabla}(\hat{T}+T)-\kappa_s\hat{\rho}\hat{T}\boldsymbol{\nabla}(\hat{s}+s), \qquad \text{by equation (B.11)} \qquad \text{(B.145)}
\end{aligned}$$

the advection term is

$$\begin{aligned}
S_{\mathrm{ADV}} &= -\hat{\rho}\hat{T}(\boldsymbol{u}\cdot\boldsymbol{\nabla})(\hat{s}+s) \\
&= -\hat{\rho}\hat{T}\left(u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)(\hat{s}+s), \qquad \text{(B.146)}
\end{aligned}$$

and the viscous heating term is

$$\begin{aligned}
S_{\mathrm{DIFF}} &= \hat{\Phi} \\
&= 2\hat{\rho}\nu_{\mathrm{eff}}\left[\underline{\underline{\boldsymbol{e}}}:\underline{\underline{\boldsymbol{e}}}-\frac{1}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{u})^2\right]. \qquad \text{by equation (B.13)} \qquad \text{(B.147)}
\end{aligned}$$