CHAPTER 5:

AN OPERATOR APPROACH TO TWO-BEAM COUPLING:

THE 2-BY-2 COMPLEX-COUPLING CASE

5.1. INTRODUCTION

In this chapter we present an operator theory for describing the process of two-beam coupling. This photorefractive interaction is represented by a black box which operates over the input optical fields to produce the output fields. The input and output fields are represented by vectors, the "fieldvectors", and the black box transformation by the coupling operator T:

$$E_{out} = T\, E_{in}\ . \qquad [5.1]$$

The operator theory is based on the same underlying physics as the standard approach for two-beam coupling. [Kukhtarev, '79] However, the nature of operators confers to this theory its main advantage over the standard approach: a mathematical notation that can deal with complicated spatial modes. This lends the power of representing the information as is the most convenient for the information processing application in hand. For example, if the input optical signals are faces, in the standard approach the faces are decomposed into a superposition of plane-waves, and a differential equation would result for each pair of plane waves, resulting in a complicated problem. With the operator theory the spatial basis may be chosen to be a set of orthogonal facial features, bridging the gap between the mathematical notation and the information processing task at hand.

Because the operator theory is based on the same physical model as the standard approach, it shares the same limitations and approximations. In particular, this theory assumes steady-state gratings, in other words, the spatial and temporal variations of the fields must be such that the gratings reach a stationary state. This means that the temporal components should vary fast enough such that a component in one beam only writes a grating with the corresponding component in the other beam, and with no other. At the other end, the temporal variations should not vary as fast as to cause the Bragg degeneracy between different optical frequencies to be broken. As mentioned earlier in this thesis, for barium titanate, these conditions limit the frequency components to lie between about hundreds of hertz and a few gigahertz for a 1cm crystal. This theory also assumes that the different spatial components are Bragg distinct, in which case we can think of each spatial component as having its own independent grating. Finally, the operator theory assumes that the coupling between the beams in each port are all described by the same coupling constant Γ. This coupling constant is in general complex, meaning that both energy or phase exchange may occur between the coupled beams. [Yeh, '93]

$$\Gamma = |\Gamma| e^{i\gamma} . \qquad [5.2]$$

The fieldvectors in equation [5.1] has two components, E_+ and E_-, corresponding respectively to the gain and loss port of the two-beam coupling interaction. In their turn E_+ and E_- can be decomposed into their various spatial modes. In general, the number of orthogonal spatial modes at the gain port, *p*, may be different of that in the loss port, *m*.

$$E = \begin{pmatrix} E_+ \\ E_- \end{pmatrix} = \begin{pmatrix} E_{+,1} \\ \vdots \\ E_{+,p} \\ E_{-,1} \\ \vdots \\ E_{-,m} \end{pmatrix} \qquad [5.3]$$

The operator theory for the general case of *p+m* spatial modes can be solved numerically and are treated in [Anderson, '00; Anderson, '99], and its details will not be covered in this chapter. Our interest is in the simple case of $p=m=1$, meaning that there is a single spatial component in each port, resulting in a 2x2 dimensionality for the coupling matrix *T*. For this “2-by-2 case”, there exists closed-form solutions for the two-beam coupling evolution and the coupling matrix *T*.

This chapter follows with three sections. Section 5.2 introduces the general operator formalism.[1] Section 5.3 treats the 2-by-2 case with complex coupling culminating with the solution of *T* for this case. We also present the special cases of purely real and purely imaginary coupling.

[1] The general theory was developed by Dr. Dana Anderson.

Back in chapter 4, section 4.4, we presented a simplified geometrical picture of the operator approach for the 2-by-2 case, assuming purely real coupling and real input fields. Although the current chapter stands alone, the simplified picture presented in chapter 4 may help lend a preliminary intuitive understanding of the operator algebra. Other approaches using an operator or matrix formulation for two-beam coupling are given in[Liu, '93; Ringhofer, '00; Stojkov, '92].

5.2. THE OPERATOR FORMULATION

Given spatially and temporally varying beams as the input to the two-beam coupling, the fields can be decomposed into a spatial basis, as shown in equation [5.3], as well as into a temporal basis. We will label the different temporal components of the input beams by the subscript ω, but it is understood that these components are not necessarily harmonic components, but any orthogonal temporal basis. The evolution of each temporal component as it propagates through the crystal is given by equation [5.1], rewritten here as

$$E_{\omega}(z) = T(z)\, E_{\omega}(0)\,, \qquad [5.4]$$

where z, the interaction length in the direction of propagation, is measured in units of $1/|\Gamma|$,where $|\Gamma|$ is the coupling constant modulus, so the only part of the coupling constant that appears in our equations is it’s phase factor γ as defined in equation

[5.2]. The field values at any position z within the crystal is determined by solving for the coupling operator T. As we will see later, T corresponds to a rotation operator which conserves the length of the fieldvectors (meaning that the transformation is lossless) and is therefore Unitary, i.e. $T^{\dagger} = T^{-1}$. The formulation involved in solving for T follows.

In an analogy to quantum mechanics, we introduce a "Hamiltonian" H, a transformation that determines the evolution of the fieldvector such that

$$E'(z) = -iH(z)\,E(z), \tag{5.5}$$

where the prime denotes derivation with respect to z. The Hamiltonian operator represents the effect of the grating on the evolution of the fields in each location z. It will later be defined such as to embody the two-beam coupling physics in the context of our operator formulation, but because the grating itself is formed by the fields, we can anticipate that H(z) will depend on the fieldvectors. However, this fact does not affect the validity of equation [5.5].

The black-box view, given by the coupling operator T, and the above fieldvector evolution, given by H, can be connected by simply substituting equation [5.4] in the above equation, giving

$$T'(z)E_{\omega}(0) = -iH(z)T(z)E_{\omega}(0), \tag{5.6}$$

which, by eliminating $E_\omega(0)$, gives us a differential equation for the coupling operator:

$$T' = -iH\,T, \qquad [5.7]$$

where the explicit z-dependence is dropped.

Our problem involves the mixing between two input beams or ports, which translates into a mixing of the components of the fieldvector. Thus a “density operator”, ρ, which represents the mixing between the components is introduced:

$$\rho = \frac{1}{I}\sum_{\omega} E_\omega \times E_\omega^\dagger, \qquad [5.8]$$

where $E_\omega^\dagger$ is the adjoint (complex-transpose) of E_ω, and I is the total intensity:

$$I = \sum_{\omega} I_\omega = \sum_{\omega} E_\omega^\dagger E_\omega .$$

In the 2-by-2 case, the density operator in the gain and loss port space is given by the density matrix:

$$\rho = \frac{1}{I}\sum_{\omega}\begin{pmatrix} |E_{\omega,+}|^2 & E_{\omega,+}E_{\omega,-}^* \\ E_{\omega,+}^*E_{\omega,-} & |E_{\omega,-}|^2 \end{pmatrix} \qquad [5.9]$$

Characterizing the system by the density operator is equivalent to characterizing it by its fieldvectors, with the advantage that it simplifies and suits the problem. [Cohen-Tannoudji, '77b] The density operator is Hermitian, and, because it is normalized, it has a unity trace:

$$Tr(\rho) = \sum_i \rho_{ii} = 1.$$

Just as we had represented the fieldvector in the black-box view of equation [5.4], we can equivalently represent the density matrix at some location *z* as a function of the input density matrix by simply substituting equation [5.4] in equation [5.8], obtaining

$$\rho = T\rho(0)T^{\dagger}. \qquad [5.10]$$

And, again, equivalently to equation [5.5] for the fieldvectors, we can write a differential equation for the density operator, which is obtained by combining the above equation [5.10] with equation [5.7], giving

$$\rho' = -iHT\rho(0)T^{\dagger} + iT\rho(0)T^{\dagger}H = -i[H,\rho], \qquad [5.11]$$

where $[A,B] = AB - BA$ is the “commutator” of *A* and *B*.

The equations above hold independently of the physics of the problem. As mentioned above the physics is included by defining an appropriate Hamiltonian, which satisfies the assumptions and approximations of the physical model described in the introduction of this chapter. It is given by

$$H = \frac{i}{4} e^{i\gamma\sigma_3} \left[\sigma_3, \rho \right], \qquad [5.12]$$

where ρ is the same density operator defined above and σ_3 is an operator which embodies the asymmetry of the coupling mechanism, and, in the 2-by-2 representation, is given by the Pauli spin matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The factor of *i* in the Hamiltonian makes it Hermitian. This two-beam coupling Hamiltonian is not derived here from first principles, but, upon substitution in equation [5.11], the resulting equations for the 2-by-2 case can be shown to reduce to that of the standard approach. [Anderson, '00]

5.3. A CLOSED-FORM SOLUTION TO THE 2-BY-2 CASE.

The operators defined in the previous section are valid in the general case, when a superposition of spatial modes are present in each beam (the n-by-n case). We now focus on the 2-by-2 case where only one spatial mode is present in each beam.

For this simple case, closed-form solutions exist for both the density operator ρ and the coupling operator *T*. From this point forward, we will cease to refer to them as operators but instead refer to their 2-by-2 matrix representation.

Any 2-by-2 matrix can be represented by a superposition of the unit matrix and the Pauli spin matrices. As we progress on the problem of deriving the coupling matrix, we will see that such a representation proves to be very well suited for the density matrix [Cohen-Tannoudji, '77a]:

$$\rho = \frac{1}{2} + \sum_{i=1}^{3} s_i \sigma_i, \qquad [5.13]$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$s_i = \frac{1}{2} Tr\left(\rho \sigma_i\right). \qquad [5.14]$$

The Pauli spin matrices have several interesting properties. First, they are idempotent, meaning that $\sigma_i^2 = 1$. Thus $e^{ia\sigma_i} = \cos a + i\sigma_i \sin a$, where a is a scalar. Second, they are cyclic upon multiplication, i.e., $\sigma_i \sigma_j = i\varepsilon_{ijk}\sigma_k$ where $i \neq j \neq k$ and ε_{ijk} is 1 for an even permutation of i,j,k and –1 for an odd permutation. Third, they anti-commute, meaning that $\{\sigma_i,\sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 0$ where $i \neq j$.

The coefficients of the spin matrices can be conveniently expressed in spherical coordinates:

$$\begin{aligned} s_1 &= \Lambda \sin(2\theta)\cos(2\phi) \\ s_2 &= \Lambda \sin(2\theta)\sin(2\phi) \\ s_3 &= \Lambda \cos(2\theta) \end{aligned} \qquad [5.15]$$

where $\Lambda = \sqrt{s_1^2 + s_2^2 + s_3^2} = (\lambda_1 - \lambda_2)/2$ where λ_1 and λ_2 are the eigenvalues of ρ, with $\lambda_1 > \lambda_2$. The ρ-matrix in the spherical coordinates is simply given by:

$$\rho = \begin{pmatrix} \frac{1}{2} + \Lambda\cos(2\theta) & \Lambda\sin(2\theta)e^{-i2\phi} \\ \Lambda\sin(2\theta)e^{i2\phi} & \frac{1}{2} - \Lambda\cos(2\theta) \end{pmatrix}. \qquad [5.16]$$

We prefer to write the solution of the density matrix and coupling matrix in terms of the spin matrices:

$$\begin{aligned} \sigma_{//} &= \sigma_1\cos(2\phi) + \sigma_2\sin(2\phi) = \begin{pmatrix} 0 & e^{-i2\phi} \\ e^{i2\phi} & 0 \end{pmatrix} \\ \sigma \; &= \sigma_2\cos(2\phi) - \sigma_1\sin(2\phi) = i\begin{pmatrix} 0 & -e^{-i2\phi} \\ e^{i2\phi} & 0 \end{pmatrix} \end{aligned} \qquad [5.17]$$

As can be deduced from the expressions above, are orthogonal and are obtained by rotating σ_1 and σ_2 around the σ_3 "axis", such that $\sigma_{//}$ is aligned with the projection of ρ into the $\sigma_1\sigma_2$-plane, as shown in figure 5.1.

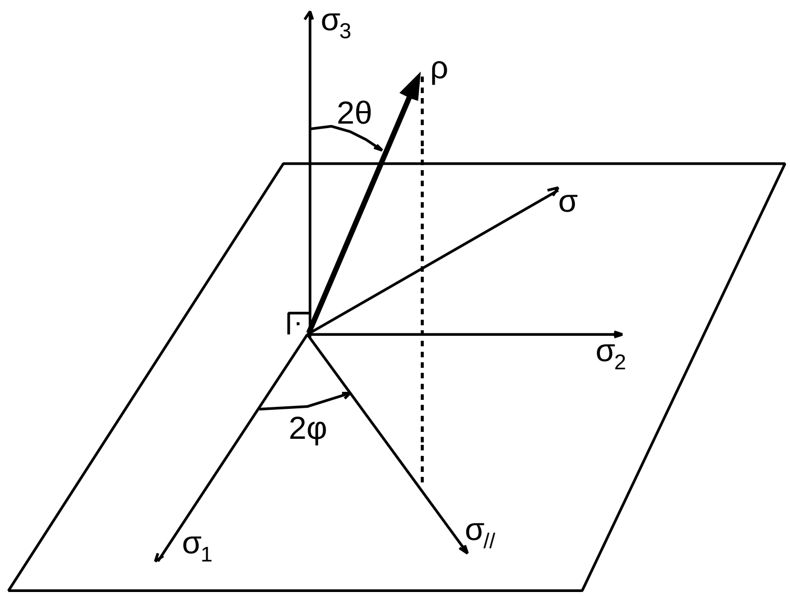


Figure 5.1. Representation of ρ in $\sigma_1\sigma_2\sigma_3$-coordinate space. The "vectors" $\sigma_{//}$ and σ are the parallel and orthogonal projection of ρ on the $\sigma_1\sigma_2$-plane and they rotate in the plane as ρ rotates in space.

Using these, we can rewrite the density matrix as:

$$\rho = \frac{1}{2} + \Lambda\sigma_{\rho} \qquad [5.18]$$

where

$$\sigma_\rho = \sigma_3 e^{i2\theta\sigma} = \sigma_{//}\sin(2\theta) + \sigma_3\cos(2\theta),$$

points in the direction of ρ.

The expression for the Hamiltonian is obtained by substituting the above ρ in equation [5.12], giving:

$$H = -\frac{\Lambda}{2}\sin(2\theta)\left(\sigma \quad \cos\gamma + \sigma_{//}\sin\gamma\right). \qquad [5.19]$$

Substituting the above H and ρ in equation [5.11], provides the differential equations for θ and ϕ:

$$\begin{aligned} \theta' &= -\frac{\Lambda}{2}\cos\gamma\sin(2\theta) \\ \phi' &= \frac{\Lambda}{2}\sin\gamma\cos(2\theta) \end{aligned} \qquad [5.20]$$

The solution for θ is straightforward:

$$\theta = \arctan\left(e^{-\Lambda z\cos\gamma}\tan\theta_0\right), \qquad [5.21]$$

where $\theta_0=\theta(0)$ and the same convention will be used from this point forward with all variables. Solving ϕ by using $\phi' = \frac{\partial\phi}{\partial\theta}\theta'$, we get

$$\phi = \frac{\tan\gamma}{2}\ln\left(\frac{\sin(2\theta_0)}{\sin(2\theta)}\right)+\phi_0 \qquad [5.22]$$

where

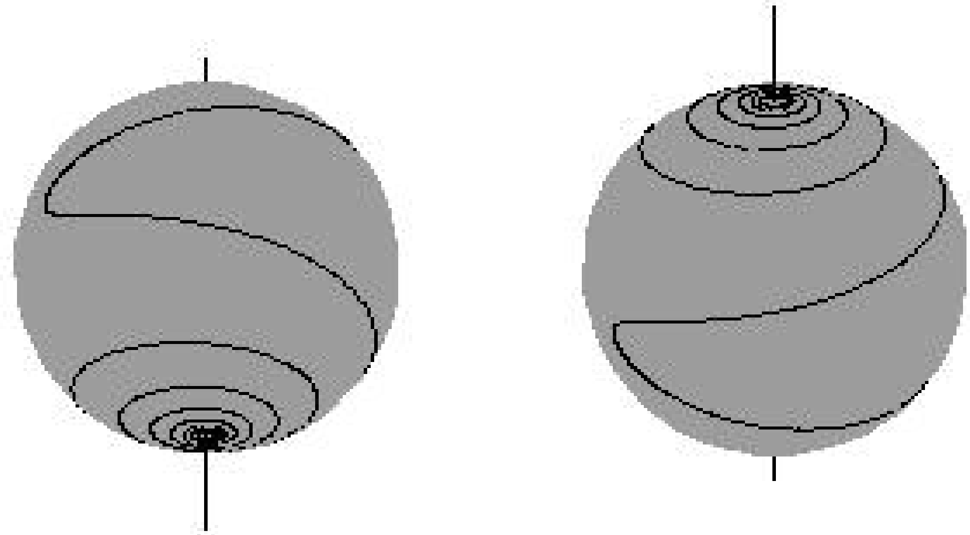
$$\sin(2\theta) = \frac{\sin(2\theta_0)}{e^{\Lambda z\cos\gamma}\cos^2\theta_0 + e^{-\Lambda z\cos\gamma}\sin^2\theta_0}$$

$$\cos(2\theta) = \frac{e^{\Lambda z\cos\gamma}\cos^2\theta_0 - e^{-\Lambda z\cos\gamma}\sin^2\theta_0}{e^{\Lambda z\cos\gamma}\cos^2\theta_0 + e^{-\Lambda z\cos\gamma}\sin^2\theta_0} \qquad [5.23]$$

The solution for ϕ is more usefully casted as:

$$e^{\pm i2\phi} = \left(\frac{\sin(2\theta_0)}{\sin(2\theta)}\right)^{\pm i\tan\gamma} e^{\pm i2\phi_0} = \left(e^{\Lambda z\cos\gamma}\cos^2\theta_0 + e^{-\Lambda z\cos\gamma}\sin^2\theta_0\right)^{\pm i\tan\gamma} e^{\pm i2\phi_0} \qquad [5.24]$$

With the solution of θ and ϕ at hand, ρ is now solved. The plot in figure 5.2 shows the shape for the evolution of ρ in the $\sigma_1\sigma_2\sigma_3$-space (as shown in figure 5.1) for γ=1.5, i.e., $0<\gamma<\pi/2$ Figure 5.2 (a) shows a view of the south pole, whereas figure 5.2 (b) shows the north pole, which points in the direction of σ_3. Given the initial conditions θ_0 and ϕ_0, ρ_0 will point to a particular point in the curve and then, as it propagates through the crystal, ρ will follow the curve asymptotically towards the north pole. The value of 2θ is an expression of the intensity ratio between the beams. The north pole corresponds to all the energy being present at the gain port, and the south pole, to it all being present at the loss port. Thus, naturally, the equator corresponds to equal intensities between the two beams. On the other hand, 2ϕ corresponds to the phase-difference between the two beams and the spiraling indicates a phase coupling between the beams. As indicated by the $\cos(2\theta)$ factor in equation [5.20] for ϕ, the spiraling in the northern hemisphere is in the opposite direction as that in the southern hemisphere. At the equator, no spiraling can occur as the curve is perpendicular to the equator line. This has a simple physical interpretation. Unlike energy transfer, two-beam coupling phase transfer is symmetric, and thus, when the beams have equal intensity they both change their phases at equal rates, resulting in no net change in the phase difference between the two beams.



(a) (b)

Figure 5.2. Plot of the evolution of the density matrix ρ as a function of propagation *z* through the medium, for γ=1.5. The initial conditions are θ_0=1.5, close to the south pole (a), where all the energy is in the loss port, and ϕ_0=0. As the density matrix evolves, it spirals up and switches direction at the equator, where θ' is maximum ϕ'=0, and then asymptotically spirals up to the north pole (b).

So, in summary, for the general case of complex coupling, where $0<\gamma<\pi/2$, the energy coupling causes ρ to approach the north pole whereas the phase coupling causes it to rotate around the polar axis σ_3, resulting in the spiral shape shown in figure 5.2. We know from photorefractive physics that a real coupling constant, which occurs for $\gamma=0$ (see equation [5.2]), corresponds to pure energy coupling and thus, for this special "real case" we expect the curve to go directly from the south to the north pole without spiraling. That is indeed the case and is shown in figure 5.3 (a), plotted for $\gamma=0$. Conversely, an imaginary coupling constant, which occurs for $\gamma=\pi/2$, corresponds to pure phase coupling and is represented by a pure rotation around σ_3. We call this the "imaginary case" and it is shown in figure 5.3 (b). Naturally, the longitude of the curve in the real case is given by the constant $\phi=\phi_0$ and similarly, the latitude of the curve in the imaginary case, by θ_0. In the real case the equation for θ is simply given by substituting $\gamma=0$ in equation [5.21]. In the imaginary case, the complex-case expression for ϕ is not valid as it becomes undetermined at that limit. Notice that when $\gamma \to \pi/2$ is plugged in equation [5.22], since $\tan(\gamma) \to \infty$ and $\ln(1) \to 0$, ϕ is undetermined. However, since in this case $\theta=\theta_0$ is constant, the expression for ϕ directly follows from its differential equation [5.20], giving $\phi(z) = \frac{\Delta z}{2}\cos(2\theta_0) + \phi_0$.

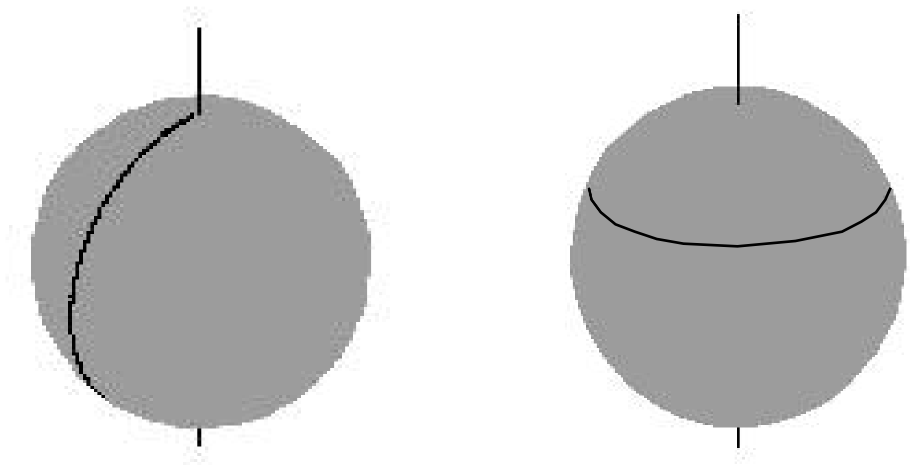


Figure 5.3. Evolution of the density matrix for the special cases of (a) real and (b) imaginary coupling. In the real case of pure energy coupling, ρ follows a longitude line towards the north pole, the longitude being determined by ϕ_0 . In the imaginary case, ρ rotates around the pole with constant speed, the latitude being determined by θ_0.

As described above, the solution to the density matrix provides us with an important tool for understanding the behavior of the system, but our ultimate goal is to solve for the coupling matrix which represents the two-beam coupling black-box. If the chain rule is applied to the differential equation for the coupling matrix, equation [5.7], and the expressions for θ', ϕ', and H (equations [5.20] and [5.19] respectively) are substituted, we get:

$$\frac{\partial T}{\partial \theta}\theta' + \frac{\partial T}{\partial \phi}\phi' + \frac{\partial T}{\partial z} = i\frac{\Lambda}{2}\sin(2\theta)\left(\sigma \quad \cos\gamma + \sigma_{//}\sin\gamma\right)T \qquad [5.25]$$

which has the following solution for *T*:

$$T = T_\theta\, T_\phi\, T_\rho = e^{-i(\theta-\theta_0)\sigma}\; e^{-i(\phi-\phi_0)\sigma_3}\, e^{i\frac{\Delta z}{2}\sin\gamma\,\sigma_{\rho 0}}. \qquad [5.26]$$

The special limit cases are straightforward by substituting $\phi=\phi_0$ and $\gamma=0$ for the real case, and $\theta=\theta_0$ and $\gamma=\pi/2$ for the imaginary case:

$$T_{\mathrm{Re}} = e^{-i(\theta-\theta_0)\sigma} \quad \text{and} \quad T_{\mathrm{Im}} = e^{-i(\phi-\phi_0)\sigma_3}\, e^{i\frac{\Delta z}{2}\sigma_{\rho 0}} = e^{-i\frac{\Delta z}{2}\cos(2\theta_0)\sigma_3}\, e^{i\frac{\Delta z}{2}\sigma_{\rho 0}}.$$

The action of T on ρ is simple: each factor of the type $T = e^{iA\sigma_i}$ corresponds to a rotation of ρ around σ_i by an amount A. So the complex solution in equation [5.26] represents a general rotation of the density matrix decomposed into the following rotations: the first factor to act on ρ, T_ρ, rotates ρ around itself, a degenerate rotation which does not affect ρ; the second factor T_ϕ then rotates ρ around σ_3 by $\phi-\phi_0$; and finally the last term T_θ rotates ρ around σ by $\theta-\theta_0$, which is a rotation directly up towards the north pole (see figure 5.1).

The interpretation of the factors T_ϕ and T_θ are clear, the first represent the relative phase difference between the beams picked up by the phase transfer, whereas the latter represents the energy coupling. However the interpretation of T_ρ is somewhat subtle. It is interesting to note that if this factor is removed from the solution, the resulting expression still satisfies the relation $\rho = T\rho(0)T^\dagger$ however it will not satisfy the differential equation $T' = -iH\,T$. We believe this is because ρ doesn't have any information about the evolution of the absolute phases of the fields, but only the relative phase between them, so it is indifferent to T_ρ. However, the two-beam coupling equations provide information about how much phase is being transferred from each beam to each other beam, and not only the difference. This information is therefore contained in *T* by this degenerate-rotation of ρ. For example, we mentioned above that ϕ' is always zero at the equator, however that does not mean that the phase transfer stops at that location, it just means that the phase transfer is symmetrical, and this information is contained in *T* by T_ρ.