Appendix A

Calculating the Shape Functions

Equation 4.5 shows the relationship between the redshift space correlation function (ξ^s) and its real space counterpart (ξ^r). For current galaxy catalogs (and those in preparation) ξ^r is not directly available. Therefore we must find a way to extract β from the properties of ξ^s. Hamilton (1993) did this by weighting the calculations of ξ by the Legendre polynomials of μ. To find the equivalent functions for the spherically symmetric case, expand ξ^s and ξ^r into sets of orthogonal components. This will allow the general functions ξ to be replaced by specific functions. Then the distortion operator will operate on a set of specific functions. The results from the distortion operation can then be added together to yield a final result for the general ξ.

One can expand in any complete set of functions but the best choice is to find a set which makes the calculations simplest. The operator $\partial/\partial \ln r_{12} \mid_{\triangle}$ commutes with all of the individual operators contained within the distortion operator (α is the only component of the distortion operator which does not necessarily commute with $\partial/\partial \ln r_{12} \mid_{\triangle}$). Here $\triangle$ is the shape of the triangle connecting the observer to the two galaxies in question. Therefore the logical choice for the expansion is the set of eigenfunctions of $\partial/\partial \ln r_{12} \mid_{\triangle}$:

$$r_{12}^{-\gamma - i\omega_{12}}. \tag{A.1}$$

Therefore, define the quantities $\xi^r(\omega_{12})$ and $\xi^s(\omega_{12})$:

$$\begin{aligned} \xi^s(\omega_{12}, \triangle) &= \frac{1}{2\pi}\int_0^\infty \xi^s(r_{12}, \triangle) r_{12}^{\gamma + i\omega_{12}} \mathrm{d}r_{12}/r_{12} \mid_{\triangle}, \\ \xi^r(\omega_{12}) &= \frac{1}{2\pi}\int_0^\infty \xi^r(r_{12}) r_{12}^{\gamma + i\omega_{12}} \mathrm{d}r_{12}/r_{12} \mid_{\triangle} \end{aligned} \tag{A.2}$$

with the inverse transforms:

$$\xi^s(r_{12},\triangle) = \int_{-\infty}^{\infty} \xi^s(\omega_{12},\triangle) r_{12}^{-\gamma-i\omega_{12}} \mathrm{d}\omega_{12},$$
$$\xi^r(r_{12}) = \int_{-\infty}^{\infty} \xi^r(\omega_{12}) r_{12}^{-\gamma-i\omega_{12}} \mathrm{d}\omega_{12}. \tag{A.3}$$

Now applying the distortion operator on $\xi^r(r_{12})$ actually applies it to the $r_{12}^{-\gamma-i\omega}$ inside the integral. This gives:

$$\xi^s(r_{12}) = \int_{-\infty}^{\infty} \xi^r(\omega_{12}) D_{\mathrm{dist}} \left[r_{12}^{-\gamma-i\omega_{12}} \right] \mathrm{d}\omega_{12} =$$
$$\int_{-\infty}^{\infty} \xi^r(\omega_{12}) \left[1 + A_1(\eta_{12},\triangle)\beta + A_2(\eta_{12},\triangle)\beta^2 \right] r_{12}^{-\gamma-i\omega_{12}} \mathrm{d}\omega_{12}. \tag{A.4}$$

Let $\eta = \gamma + i\omega$. Now calculate the actual values of each A.

To calculate the values of A operate with the distortion operator on $r^{-\eta}$. First of all, note that the operators with subscripts 1 commute with those with subscripts 2. So

$$A_1(\eta_{12},\triangle) r^{-\eta} = \left[(\frac{\partial^2}{\partial r_1^2} + \frac{\alpha(r_1)}{r_1}\frac{\partial}{\partial r_1}) \nabla_1^{-2} + (\frac{\partial^2}{\partial r_2^2} + \frac{\alpha(r_2)}{r_2}\frac{\partial}{\partial r_2}) \nabla_2^{-2} \right] r_{12}^{-\eta} \tag{A.5}$$

and

$$A_2(\eta_{12},\triangle) r^{-\eta} = \left[(\frac{\partial^2}{\partial r_1^2} + \frac{\alpha(r_1)}{r_1}\frac{\partial}{\partial r_1})(\frac{\partial^2}{\partial r_2^2} + \frac{\alpha(r_2)}{r_2}\frac{\partial}{\partial r_2}) \nabla_1^{-2} \nabla_2^{-2} \right] r_{12}^{-\eta}. \tag{A.6}$$

The first job here is to operate with ∇_i^{-2} on $r_{12}^{-\eta}$. Because r_{12} is a spherically symmetric quantity:

$$\nabla_1^{-2} r_{12}^{-\eta} = \nabla_2^{-2} r_{12}^{-\eta} = \left[\frac{1}{r_{12}^2} \frac{\partial}{\partial r_{12}} r_{12}^2 \frac{\partial}{\partial r_{12}} \right]^{-1} r_{12}^{-\eta}. \tag{A.7}$$

Set this equal to a new variable ϕ and solve for ϕ:

$$\left[\frac{1}{r_{12}^2} \frac{\partial}{\partial r_{12}} r_{12}^2 \frac{\partial}{\partial r_{12}} \right] \phi = r_{12}^{-\eta} \tag{A.8}$$

becomes

$$\left[\partial (r_{12}^2 \frac{\partial}{\partial r_{12}}) \right] \phi = r_{12}^{2-\eta} \partial r_{12}. \tag{A.9}$$

After performing one integration

$$\left[r_{12}^2\frac{\partial}{\partial r_{12}}\right]\phi=\frac{r_{12}^{3-\eta}}{3-\eta}. \tag{A.10}$$

or

$$\partial\phi=\frac{r_{12}^{1-\eta}}{3-\eta}\partial r_{12}. \tag{A.11}$$

Performing this integration yields

$$\phi=\frac{r_{12}^{2-\eta}}{(3-\eta)(2-\eta)}. \tag{A.12}$$

Note that this equation serves for any η provided that η does not equal 2 or 3. This means that

$$\nabla_1^{-2}r_{12}^{-\eta}=\nabla_2^{-2}r_{12}^{-\eta}=\frac{r_{12}^{2-\eta}}{(3-\eta)(2-\eta)} \tag{A.13}$$

and by considering $-\eta'=2-\eta$:

$$\nabla_1^{-2}\nabla_2^{-2}r_{12}^{-\eta}=\frac{r_{12}^{4-\eta}}{(5-\eta)(4-\eta)(3-\eta)(2-\eta)}. \tag{A.14}$$

Now the values for A become

$$A_1(\eta_{12},\triangle)r^{-\eta} \;=\; \left[(\frac{\partial^2}{\partial r_1^2}+\frac{\alpha(r_1)}{r_1}\frac{\partial}{\partial r_1})+(\frac{\partial^2}{\partial r_2^2}+\frac{\alpha(r_2)}{r_2}\frac{\partial}{\partial r_2})\right]\frac{r_{12}^{2-\eta}}{(3-\eta)(2-\eta)} \tag{A.15}$$

and

$$A_2(\eta_{12},\triangle)r^{-\eta}= \left[(\frac{\partial^2}{\partial r_1^2}+\frac{\alpha(r_1)}{r_1}\frac{\partial}{\partial r_1})(\frac{\partial^2}{\partial r_2^2}+\frac{\alpha(r_2)}{r_2}\frac{\partial}{\partial r_2})\right]\frac{r_{12}^{4-\eta}}{(5-\eta)(4-\eta)(3-\eta)(2-\eta)}. \tag{A.16}$$

To calculate the results for A_1 and A_2 start by doing some preliminary calculations. First define the cosines of the angles in the triangle:

$$\mu_1=\frac{r_{12}^2+r_1^2-r_2^2}{2r_{12}r_1},\mu_2=\frac{r_{12}^2+r_2^2-r_1^2}{2r_{12}r_2},\mu_{12}=\frac{r_1^2+r_2^2-r_{12}^2}{2r_1r_2}. \tag{A.17}$$

This means that

$$r_{12}=\left[r_1^2+r_2^2-2r_1r_2\mu_{12}\right]^{\frac{1}{2}}. \tag{A.18}$$

So,

$$\frac{\partial}{\partial r_1} r_{12}^n = n r_{12}^{n-2}(r_1 - r_2\mu_{12}) \tag{A.19}$$

and

$$\frac{\partial}{\partial r_2} r_{12}^n = n r_{12}^{n-2}(r_2 - r_1\mu_{12}). \tag{A.20}$$

Simplify the terms in parentheses by noting

$$r_1 - r_2\mu_{12} = \frac{2r_1^2 r_2 - r_2(r_1^2 + r_2^2 - r_{12}^2)}{2r_1 r_2} = \frac{r_1^2 - r_2^2 + r_{12}^2}{2r_1} = \mu_1 r_{12}, \tag{A.21}$$

and similarly

$$r_2 - r_1\mu_{12} = \mu_2 r_{12}. \tag{A.22}$$

Another useful equation is

$$2\mu_1\mu_2\mu_{12} + \mu_1^2 + \mu_2^2 + \mu_{12}^2 = 1. \tag{A.23}$$

Now divide A_1 into two pieces. The first piece has α terms only:

$$\left[\frac{\alpha(r_1)}{r_1}\frac{\partial}{\partial r_1} + \frac{\alpha(r_2)}{r_2}\frac{\partial}{\partial r_2}\right]\frac{r_{12}^{2-\eta}}{(3-\eta)(2-\eta)}. \tag{A.24}$$

Differentiating gives:

$$\left[\frac{\alpha(r_1)}{r_1} r_{12}\mu_1 + \frac{\alpha(r_2)}{r_2} r_{12}\mu_2\right]\frac{r_{12}^{-\eta}}{3-\eta}. \tag{A.25}$$

The second term has no terms with α:

$$\left[\frac{\partial^2}{\partial r_1^2} + \frac{\partial^2}{\partial r_2^2}\right]\frac{r_{12}^{2-\eta}}{(3-\eta)(2-\eta)} = \left[1 - \eta\mu_1^2 + 1 - \eta\mu_2^2\right]\frac{r_{12}^{-\eta}}{3-\eta}. \tag{A.26}$$

Later in the calculation, all factors of η will need to be in the denominator. So by noticing that

$$\frac{-\eta}{3-\eta} = 1 - \frac{3}{3-\eta} \tag{A.27}$$

A_1 can be written:

$$A_1(\eta,\triangle)r_{12}^{-\eta} = \left[\frac{\frac{\alpha(r_1)}{r_1}r_{12}\mu_1 + \frac{\alpha(r_2)}{r_2}r_{12}\mu_2 + 2 - 3\mu_1^2 - 3\mu_2^2}{3-\eta} + \mu_1^2 + \mu_2^2\right] r_{12}^{-\eta}. \tag{A.28}$$

To attack the differentiation necessary for A_2 first notice that it can be rewritten:

$$\begin{aligned} A_2(\eta,\triangle)r_{12}^{-\eta} = \left[(\frac{\partial}{\partial r_1} + \frac{\alpha(r_1)}{r_1})(\frac{\partial}{\partial r_2} + \frac{\alpha(r_2)}{r_2})\frac{\partial}{\partial r_1}\frac{\partial}{\partial r_2}\right] \\ \frac{r_{12}^{4-\eta}}{(5-\eta)(4-\eta)(3-\eta)(2-\eta)}. \end{aligned} \tag{A.29}$$

First differentiating with respect to r_2 yields:

$$A_2(\eta,\triangle)r_{12}^{-\eta} = \left[(\frac{\partial}{\partial r_1} + \frac{\alpha(r_1)}{r_1})(\frac{\partial}{\partial r_2} + \frac{\alpha(r_2)}{r_2})\frac{\partial}{\partial r_1}\right] \frac{(r_2 - r_1\mu_{12})r_{12}^{2-\eta}}{(5-\eta)(3-\eta)(2-\eta)}. \tag{A.30}$$

After the second differentiation:

$$\begin{aligned} A_2(\eta,\triangle)r_{12}^{-\eta} = \left[(\frac{\partial}{\partial r_1} + \frac{\alpha(r_1)}{r_1})(\frac{\partial}{\partial r_2} + \frac{\alpha(r_2)}{r_2})\right] \\ \left[\frac{-\mu_{12}r_{12}^{2-\eta}}{(5-\eta)(3-\eta)(2-\eta)} + \frac{(r_1 - r_2\mu_{12})(r_2 - r_1\mu_{12})r_{12}^{-\eta}}{(5-\eta)(3-\eta)}\right]. \end{aligned} \tag{A.31}$$

Once again separate out the different terms and calculate them separately to obtain the simplest looking results. The first term is the term with two factors of α:

$$\frac{\alpha(r_1)}{r_1}\frac{\alpha(r_2)}{r_2}\left[\frac{-\mu_{12}r_{12}^{2-\eta}}{(5-\eta)(3-\eta)(2-\eta)} + \frac{\mu_1\mu_2 r_{12}^{2-\eta}}{(5-\eta)(3-\eta)}\right]. \tag{A.32}$$

The second term has one factor of α:

$$\begin{aligned} \Bigg[\frac{\alpha(r_1)}{r_1}\frac{-2\mu_{12}\mu_2 r_{12} + \mu_1 r_{12} - \eta\mu_1\mu_2^2 r_{12}}{(5-\eta)(3-\eta)} + \\ \frac{\alpha(r_2)}{r_2}\frac{-2\mu_{12}\mu_1 r_{12} + \mu_2 r_{12} - \eta\mu_1^2\mu_2 r_{12}}{(5-\eta)(3-\eta)}\Bigg] r_{12}^{-\eta}. \end{aligned} \tag{A.33}$$

The final term contains no factors of α:

$$\begin{aligned} \frac{\partial}{\partial r_1}\Bigg[\frac{-2\mu_{12}(r_2 - r_1\mu_{12})r_{12}^{-\eta}}{(5-\eta)(3-\eta)} + \frac{(r_1 - r_2\mu_{12})r_{12}^{-\eta}}{(5-\eta)(3-\eta)} \\ - \frac{\eta(r_1 - r_2\mu_{12})(r_2 - r_1\mu_{12})^2 r_{12}^{-2-\eta}}{(5-\eta)(3-\eta)}\Bigg] = \\ \frac{2\mu_{12}^2 r_{12}^{-\eta}}{(5-\eta)(3-\eta)} + \frac{4\eta\mu_{12}\mu_1\mu_2 r_{12}^{-\eta}}{(5-\eta)(3-\eta)} + \frac{r_{12}^{-\eta}}{(5-\eta)(3-\eta)} + \\ \frac{-\eta\mu_1^2 r_{12}^{-\eta}}{(5-\eta)(3-\eta)} + \frac{-\eta\mu_2^2 r_{12}^{-\eta}}{(5-\eta)(3-\eta)} + \frac{\eta(2+\eta)\mu_1^2\mu_2^2 r_{12}^{-\eta}}{(5-\eta)(3-\eta)}. \end{aligned} \tag{A.34}$$

Using equation A.23 rewrite this term:

$$\begin{aligned}
&\frac{2\mu_{12}^2 r_{12}^{-\eta}}{(5-\eta)(3-\eta)} + \frac{-2\eta(\mu_{12}^2+\mu_1^2+\mu_2^2-1)r_{12}^{-\eta}}{(5-\eta)(3-\eta)} + \frac{r_{12}^{-\eta}}{(5-\eta)(3-\eta)} + \\
&\qquad \frac{-\eta\mu_1^2 r_{12}^{-\eta}}{(5-\eta)(3-\eta)} + \frac{-\eta\mu_2^2 r_{12}^{-\eta}}{(5-\eta)(3-\eta)} + \frac{\eta(2+\eta)\mu_1^2\mu_2^2 r_{12}^{-\eta}}{(5-\eta)(3-\eta)} = \\
&\qquad\qquad \frac{2\mu_{12}^2(1-\eta)r_{12}^{-\eta}}{(5-\eta)(3-\eta)} + \frac{(1+2\eta)r_{12}^{-\eta}}{(5-\eta)(3-\eta)} + \\
&\qquad\qquad \frac{-3\eta\mu_1^2 r_{12}^{-\eta}}{(5-\eta)(3-\eta)} + \frac{-3\eta\mu_2^2 r_{12}^{-\eta}}{(5-\eta)(3-\eta)} + \frac{\eta(2+\eta)\mu_1^2\mu_2^2 r_{12}^{-\eta}}{(5-\eta)(3-\eta)}.
\end{aligned} \tag{A.35}$$

Each $A_i(\eta_{12}, \triangle)$ can be expressed in functions of r_{12},r_1 and r_2 times functions of $\frac{1}{n-\eta_{12}}$ where $n = 2, 3, 5$. The factors of β and the r's in equation (A.4) can be brought out of integral over ω to become:

$$\xi^s(r_{12}) = \sum_j g_j(\beta) B_j(r_{12}, r_1, r_2) \int_{-\infty}^{\infty} \frac{\xi^r(\omega_{12}) r_{12}^{-\eta_{12}}}{n_j - \eta_{12}} \mathrm{d}\omega_{12}. \tag{A.36}$$

Rearranging the integral in this equation:

$$\int_{-\infty}^{\infty} \frac{\xi^r(\omega_{12}) r_{12}^{-\eta_{12}}}{n - \eta_{12}} \mathrm{d}\omega_{12} = \int_{-\infty}^{\infty} \xi^r(\omega_{12}) \left[\frac{1}{r_{12}^n} \int_0^{r_{12}} r^{n-\eta_{12}-1} \mathrm{d}r \right] \mathrm{d}\omega_{12}, \tag{A.37}$$

and switching the order of integration:

$$\frac{1}{r_{12}^n} \int_0^{r_{12}} r^{n-1} \left[\int_{-\infty}^{\infty} \xi^r(\omega) r^{-\eta_{12}} \right] \mathrm{d}r = \frac{1}{r_{12}^n} \int_0^{r_{12}} r^{n-1} \xi^r(r) \mathrm{d}r \tag{A.38}$$

So the integral can be expressed in an average over ξ. Let:

$$\xi^s(r_{12}) = \sum_j g_j(\beta) B_j(r_{12}, r_1, r_2) \Xi_j(\xi(r_{12}), \breve{\xi}(r_{12}), \bar{\xi}(r_{12}), \bar{\bar{\xi}}(r_{12})) \tag{A.39}$$

with

$$\begin{aligned}
\breve{\xi} &\equiv \frac{2}{r_{12}^2} \int_0^{r_{12}} \xi(r) r \mathrm{d}r, \\
\bar{\xi} &\equiv \frac{3}{r_{12}^3} \int_0^{r_{12}} \xi(r) r^2 \mathrm{d}r, \\
\bar{\bar{\xi}} &\equiv \frac{5}{r_{12}^5} \int_0^{r_{12}} \xi(r) r^4 \mathrm{d}r.
\end{aligned} \tag{A.40}$$

Note that in equation A.39 the functions $B_j(r_i, r_j, r_{ij})$, $g_j(\beta)$ and f_j are not unique. Choose any set of functions which satisfy equation A.39. Therefore, choose the functions to make the rest of the analysis as simple and accurate as possible.

In order to obtain results that tend towards the distant observer approximation, choose three of the shape functions as:

$$B_0 \equiv 1, \tag{A.41}$$

$$B_2 \equiv \frac{1}{2}\left[P_2(\mu_1) + P_2(\mu_2)\right], \tag{A.42}$$

and

$$B_4 \equiv \frac{1}{8}(35\mu_1^2\mu_2^2 - 15\mu_1^2 - 15\mu_2^2 + 3) \tag{A.43}$$

where

$$P_2 = \frac{3}{2}\mu^2 - \frac{1}{2}. \tag{A.44}$$

Define $\check{\xi}_i$ as the function $g_i(\beta)\Xi_i(\xi(r_{12}), \breve{\xi}(r_{12}), \bar{\xi}(r_{12}), \bar{\bar{\xi}}(r_{12}))$ which corresponds to shape function B_i. This means that the term with no factors of β contributes $\xi(r_{12})$ to $\check{\xi}_0$. The terms given by A_1 contribute $\frac{4}{3}\beta(\xi(r_{12}) - \bar{\xi}(r_{12}))$ to $\check{\xi}_2$ and $\frac{2}{3}\beta\xi(r_{12})$ to $\check{\xi}_0$ leaving

$$\frac{\beta}{3}\bar{\xi}(r_{12})\left[\frac{\alpha(r_1)}{r_1}r_{12}\mu_1 + \frac{\alpha(r_2)}{r_2}r_{12}\mu_2\right]. \tag{A.45}$$

So, define

$$B_1 \equiv \left[\frac{\alpha(r_1)}{r_1}r_{12}\mu_1 + \frac{\alpha(r_2)}{r_2}r_{12}\mu_2\right]. \tag{A.46}$$

This means that the terms of A_1 also contributes $\frac{\beta}{3}\bar{\xi}(r_{12})$ to $\check{\xi}_1$.

The terms in equation A.35 will contribute to $\check{\xi}_4$, $\check{\xi}_2$, and $\check{\xi}_0$. To find the exact contributions, replace $\mu_1^2\mu_2^2$

$$\mu_1^2\mu_2^2 = \frac{8}{35}B_4 + \frac{3}{7}(\mu_1^2 + \mu_2^2) - \frac{3}{35} \tag{A.47}$$

to get

$$\frac{r_{12}^{-\eta}}{(5-\eta)(3-\eta)}\left[2\mu_{12}^2(1-\eta)+(1+2\eta)-3\eta(\mu_1^2+\mu_2^2)+\right.$$
$$\left.\eta(2+\eta)(\frac{8}{35}B_4+\frac{3}{7}(\mu_1^2+\mu_2^2)-\frac{3}{35})\right]=$$
$$\frac{r_{12}^{-\eta}}{(5-\eta)(3-\eta)}\left[2\mu_{12}^2(1-\eta)+(1+\frac{64}{35}\eta-\frac{3}{35}\eta^2)+\right.$$
$$\left.(\mu_1^2+\mu_2^2)(\frac{3}{7}\eta^2-\frac{15}{7}\eta)+\eta(2+\eta)\frac{8}{35}B_4\right]. \quad \text{(A.48)}$$

Now replace $\mu_1^2+\mu_2^2$ using

$$\mu_1^2+\mu_2^2=\frac{4}{3}B_2+\frac{2}{3} \quad \text{(A.49)}$$

to get

$$\frac{r_{12}^{-\eta}}{(5-\eta)(3-\eta)}\left[2\mu_{12}^2(1-\eta)+(1+\frac{64}{35}\eta-\frac{3}{35}\eta^2)+(\frac{4}{3}B_2+\frac{2}{3})(\frac{3}{7}\eta^2-\frac{15}{7}\eta)+\right.$$
$$\left.\eta(2+\eta)\frac{8}{35}B_4\right]=$$
$$\frac{r_{12}^{-\eta}}{(5-\eta)(3-\eta)}\left[2\mu_{12}(1-\eta)+(1+\frac{2}{5}\eta+\frac{1}{5}\eta^2)+\right.$$
$$\left.(\frac{3}{7}\eta^2-\frac{15}{7}\eta)\frac{4}{3}B_2+\eta(2+\eta)\frac{8}{35}B_4\right]. \quad \text{(A.50)}$$

Before continuing, note a few relations:

$$\frac{\eta(2+\eta)}{(5-\eta)(3-\eta)}=1+\frac{15}{2}\frac{1}{3-\eta}-\frac{35}{2}\frac{1}{5-\eta}\Longrightarrow\xi(r_{12})+\frac{5}{2}\bar{\xi}(r_{12})-\frac{7}{2}\bar{\bar{\xi}}(r_{12}), \quad \text{(A.51)}$$

$$\frac{\frac{3}{7}\eta^2-\frac{15}{7}\eta}{(5-\eta)(3-\eta)}=\frac{3}{7}-\frac{9}{7}\frac{1}{3-\eta}\Longrightarrow\frac{3}{7}\xi(r_{12})-\frac{3}{7}\bar{\xi}(r_{12}), \quad \text{(A.52)}$$

$$\frac{1+\frac{2}{5}\eta+\frac{1}{5}\eta^2}{(5-\eta)(3-\eta)}=\frac{1}{5}+\frac{2}{3-\eta}-\frac{4}{5-\eta}\Longrightarrow\frac{1}{5}\xi(r_{12})+\frac{2}{3}\bar{\xi}(r_{12})-\frac{4}{5}\bar{\bar{\xi}}(r_{12}), \quad \text{(A.53)}$$

and

$$\frac{1-\eta}{(5-\eta)(3-\eta)}=-\frac{1}{3-\eta}+\frac{2}{5-\eta}\Longrightarrow-\frac{1}{3}\bar{\xi}(r_{12})+\frac{2}{5}\bar{\bar{\xi}}(r_{12}). \quad \text{(A.54)}$$

The terms of β^2 which contain factors of α cannot contribute to the values of $\breve{\xi}_0$, $\breve{\xi}_2$, or $\breve{\xi}_4$ because the corresponding shape functions contain no factors of α. This means that

$$\breve{\xi}_0 = \left[1 + \frac{2}{3}\beta + \frac{1}{5}\beta^2\right] \xi(r_{12}), \tag{A.55}$$

$$\breve{\xi}_2 = \left[\frac{4}{3}\beta + \frac{4}{7}\beta^2\right] \left[\xi(r_{12}) - \bar{\xi}(r_{12})\right], \tag{A.56}$$

and

$$\breve{\xi}_4 = \frac{8}{35}\beta^2 \left[\xi(r_{12}) + \frac{5}{2}\bar{\xi}(r_{12}) - \frac{7}{2}\bar{\bar{\xi}}(r_{12})\right]. \tag{A.57}$$

This leaves the terms

$$2\beta^2 \left[-\frac{1}{3}\bar{\xi}(r_{12}) + \frac{2}{5}\bar{\bar{\xi}}(r_{12})\right] \mu_{12}^2 + \beta^2 \left[\frac{2}{3}\bar{\xi}(r_{12}) - \frac{4}{5}\bar{\bar{\xi}}(r_{12})\right] \tag{A.58}$$

and all of the terms with factors of α for some other shape function. Reduce the terms containing factors of α with

$$\frac{1}{(5-\eta)(3-\eta)(2-\eta)} = \frac{1}{3}\frac{1}{2-\eta} - \frac{1}{2}\frac{1}{3-\eta} + \frac{1}{6}\frac{1}{5-\eta} \Longrightarrow \frac{1}{6}\breve{\xi}(r_{12}) - \frac{1}{6}\bar{\xi}(r_{12}) + \frac{1}{30}\bar{\bar{\xi}}(r_{12}), \tag{A.59}$$

$$\frac{1}{(5-\eta)(3-\eta)} = \frac{1}{2}\frac{1}{3-\eta} - \frac{1}{2}\frac{1}{5-\eta} \Longrightarrow \frac{1}{6}\bar{\xi}(r_{12}) - \frac{1}{10}\bar{\bar{\xi}}(r_{12}), \tag{A.60}$$

and

$$\frac{\eta}{(5-\eta)(3-\eta)} = \frac{3}{2}\frac{1}{3-\eta} - \frac{5}{2}\frac{1}{5-\eta} \Longrightarrow \frac{1}{2}\bar{\xi}(r_{12}) - \frac{1}{2}\bar{\bar{\xi}}(r_{12}). \tag{A.61}$$

Notice that the remaining terms contain no factors of $\xi(r_{12})$ and that all of the remaining terms have a factor of β^2. The easiest solution to capturing all of the remaining terms is to define

$$\breve{\xi}_3 = \beta^2 \bar{\xi}(r_{12}), \tag{A.62}$$

$$\check{\xi}_5 = \beta^2 \bar{\bar{\xi}}(r_{12}), \tag{A.63}$$

and

$$\check{\xi}_6 = \frac{\beta^2}{6} \breve{\xi}(r_{12}). \tag{A.64}$$

These then have the correspondingly complicated shape funcitons:

$$\begin{aligned} B_3 = {} & -\frac{2}{3}\mu_{12}^2 + \frac{2}{3} + \frac{1}{6}\frac{\alpha(r_1)\alpha(r_2)\mu_{12}r_{12}^2}{r_1 r_2} + \frac{1}{6}\frac{\alpha(r_1)\alpha(r_2)\mu_1\mu_2 r_{12}^2}{r_1 r_2} - \\ & \frac{1}{2}\left[\frac{\alpha(r_1)r_{12}}{r_1}\mu_1\mu_2^2 + \frac{\alpha(r_2)r_{12}}{r_2}\mu_1^2\mu_2\right] - \frac{1}{3}\left[\frac{\alpha(r_1)r_{12}}{r_1}\mu_{12}\mu_2 + \frac{\alpha(r_2)r_{12}}{r_2}\mu_1\mu_{12}\right] \\ & \left\{+\frac{1}{6}\left[\frac{\alpha(r_1)r_{12}}{r_1}\mu_1 + \frac{\alpha(r_2)r_{12}}{r_2}\mu_2\right]\right\}, \end{aligned} \tag{A.65}$$

$$\begin{aligned} B_5 = {} & \frac{4}{5}\mu_{12}^2 - \frac{4}{5} - \frac{1}{30}\frac{\alpha(r_1)\alpha(r_2)\mu_{12}r_{12}^2}{r_1 r_2} - \frac{1}{10}\frac{\alpha(r_1)\alpha(r_2)\mu_1\mu_2 r_{12}^2}{r_1 r_2} + \\ & \frac{1}{2}\left[\frac{\alpha(r_1)r_{12}}{r_1}\mu_1\mu_2^2 + \frac{\alpha(r_2)r_{12}}{r_2}\mu_1^2\mu_2\right] + \frac{1}{5}\left[\frac{\alpha(r_1)r_{12}}{r_1}\mu_{12}\mu_2 + \frac{\alpha(r_2)r_{12}}{r_2}\mu_1\mu_{12}\right] \\ & - \frac{1}{10}\left[\frac{\alpha(r_1)r_{12}}{r_1}\mu_1 + \frac{\alpha(r_2)r_{12}}{r_2}\mu_2\right], \end{aligned} \tag{A.66}$$

$$B_6 = -\frac{\alpha(r_1)\alpha(r_2)\mu_{12}r_{12}^2}{r_1 r_2} \tag{A.67}$$

Notice that the quantity in curly braces in equation A.65 is exactly the same as B_1. Therefore, remove this term from equation A.65 and say that

$$\check{\xi}_1 = (\frac{\beta}{3} + \frac{\beta^2}{6})\bar{\xi}(r_{12}). \tag{A.68}$$

The final term of B_5 also has the form of B_1. However adding this term to $\check{\xi}_1$ would make the β term inseparable from the averages of ξ.

Although the shape functions and resulting $\check{\xi}$'s are not unique, the above set of functions is complete and has the desireable property of being seperable into functions of the shape, functions of β and functions of averages of ξ^r. This set of functions will allow for a complete description of spherical redshift distortions in the linear regime.