

Chapter 5

Chern-Simons theory for a strongly correlated rotating Bose gas

5.1 Introduction

The quasi-hole excitations in the bosonic Laughlin state are $\frac{1}{2}$ bosons as argued in the last chapter. Therefore a field theory model appropriate for the Bose gas in the strongly correlated Laughlin regime should exhibit vortex solutions corresponding to such quasi-holes. Initial ideas of a field theory appropriate to describe such a strongly correlated state were put forth by Girvin and MacDonald [100] in relation to the electronic FQHE. A related model was developed by Zhang, Hansson, and Kivelson (ZHK) [98] directly from the microscopic Hamiltonian. While the starting points were different, both these theories involved coupling of a scalar field to a gauge potential (a_0, \mathbf{a}) with a Chern-Simons action (or topological mass term). To understand the necessity of such a gauge term and its relevance to the strongly correlated Bose system that we wish to study, we first need to understand the origin of fractional statistics in a two-dimensional space.

We begin by discussing the origin of anyons, particles with statistics intermediate between Fermi and Bose statistics in two-dimensional space. In fact it is possible to change the statistics of particles in two-dimensional space, for example by attaching fictitious magnetic flux in the case of electrons. The ZHK theory uses precisely this transformation property associated with the gauge term of the Chern-Simons theory to describe the strongly correlated fractional quantum Hall state as a Bose condensed

	3D	2D
• Quantization	$[S_i, S_j] = i\hbar \varepsilon_{ijk} S_k$	S_z
• Exchange statistics	+1 (bosons) -1 (fermions)	$e^{i\alpha\pi}$

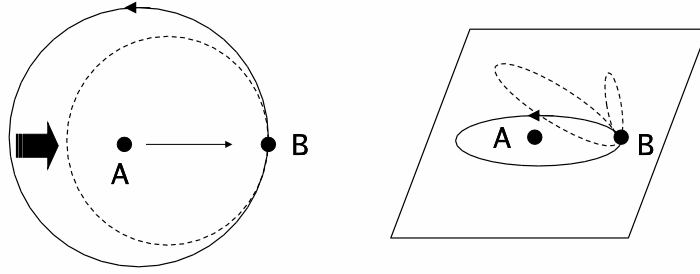


Figure 5.1: Statistics associated with the exchange of two identical particles. The net effect of making the interchange twice is to take one particle around another in a closed loop. In 3-dimensions the loop can be deformed without crossing the other particle which is not possible in two dimensions.

state of bosonized electrons (composite bosons). Within this view point, we show that the atomic Laughlin state of previous chapter can be described as a Bose condensed state of composite bosons.

5.2 Statistics in 2-dimensions

In 3-dimensional space there are only two types of particles, bosons and fermions. They are distinguished by the following properties: (a) intrinsic spin of bosons is $s\hbar$; $s = 0, 1, 2, \dots$ while that of fermions is $s\hbar$; $s = \frac{1}{2}, \frac{3}{2}, \dots$, (b) when two identical particles are exchanged, the wavefunction changes its overall sign for fermions but not for bosons, and (c) any number of bosons can occupy a single quantum state, but a single state cannot be occupied by more than one fermion. The non-Abelian algebra,

$$[S_i, S_j] = i\hbar \varepsilon_{ijk} S_k \text{ for } i, j, k = x, y, z \quad (5.1)$$

is responsible for the quantization of spin in 3-dimensions. To the contrary, fractional spin is possible in a 2-dimensional space because there is only one way of rotation, which is around the z axis by embedding the 2-dimensional space into the 3-dimensional space and identifying it with the xy plane. Here the spin operator is S_z with an arbitrary eigenvalue since the rotational group is Abelian. Now for the exchange statistics, as shown in Fig. 5.1, if we move one particle around the other in a closed loop, in 3-dimensions, the loop can be continuously deformed to a point which in 2-dimensional space is not possible without crossing the second particle. Hence in 2-dimensional space, the exchange statistics depends on the loop contrary to 3-dimensional space, where it is an intrinsic property of the particles. Thus particles in 2-dimensions can act like fermions or bosons under exchange depending on the geometry of exchange. Lastly, the Pauli exclusion principle can be interpolated from fermions to anyons (particles in two dimensions with fractional spin) by introducing a statistical interaction g [101] defined by

$$g = \frac{d_N - d_{N+\Delta N}}{\Delta N}, \quad (5.2)$$

where N is the number of particles and d_N is the dimension of the one-particle Hilbert space obtained by holding the coordinates of $N - 1$ particles fixed. We have $g = 0$ for bosons, $g = 1$ for fermions and $g = m$ for the composite particles in the $\frac{1}{m}$ fractional quantum Hall state.

5.3 Berry phase

Consider a system in which the Hamiltonian depends on some external parameter represented by a vector \mathbf{R} . Also assume that the Hamiltonian has a non-degenerate eigenstate $\Psi_{\mathbf{R}}$

$$H(\mathbf{R})\Psi_{\mathbf{R}} = E_{\mathbf{R}}\Psi_{\mathbf{R}} \quad (5.3)$$

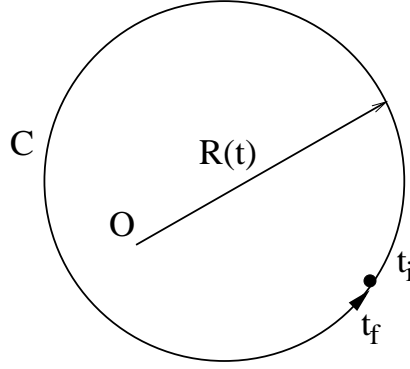


Figure 5.2: Trajectory of the vector $\mathbf{R}(t)$ in the parameter space. The vector $\mathbf{R}(t)$ represents a set of parameters that define the Hamiltonian at some time t . The parameters are changed adiabatically from their initial value at time t_i such that the vector moves in a closed loop and comes back to its original position at some final time t_f

Now we start at some initial time t_i and change the parameters such that the vector \mathbf{R} moves in a closed loop C in the parameter space as shown in Fig. 5.2 and returns to its initial value at some final time t_f , $\mathbf{R}(t_f) = \mathbf{R}(t_i)$. If the change is sufficiently slow the state changes continuously and remains an eigenstate at all times. Since we have assumed the state to be non-degenerate, the state at time t_f should be the same as that at t_i except for a multiplicative phase factor. In fact, the states are related by

$$\Psi_{\mathbf{R}(t_f)}(t_f) = \exp\left(-\frac{i}{\hbar} \int_{t_i}^{t_f} dt' E(\mathbf{R}(t')) + i\gamma(C)\right) \Psi_{\mathbf{R}(t_i)}(t_i). \quad (5.4)$$

The first part of the phase factor is the dynamical phase and is due to the eigenenergy of the state. The second part depends on the contour C and is independent of $t_f - t_i$ and is referred to as the geometric phase. This phase can be expressed as

$$\gamma(C) = i \oint_C \langle \Psi_{\mathbf{R}(t)} | \nabla_{\mathbf{R}(t)} \Psi_{\mathbf{R}(t)} \rangle \cdot d\mathbf{R}(t) \quad (5.5)$$

and is called the Berry phase [102]. A typical example is the Aharonov-Bohm (AB) phase [103], which appears when an electron moves in a magnetic field. The AB phase is given by $\gamma(C) = eBS/\hbar$, where BS is the flux passing through the contour C . In the next section we will see how this extra geometric phase is associated with the idea of

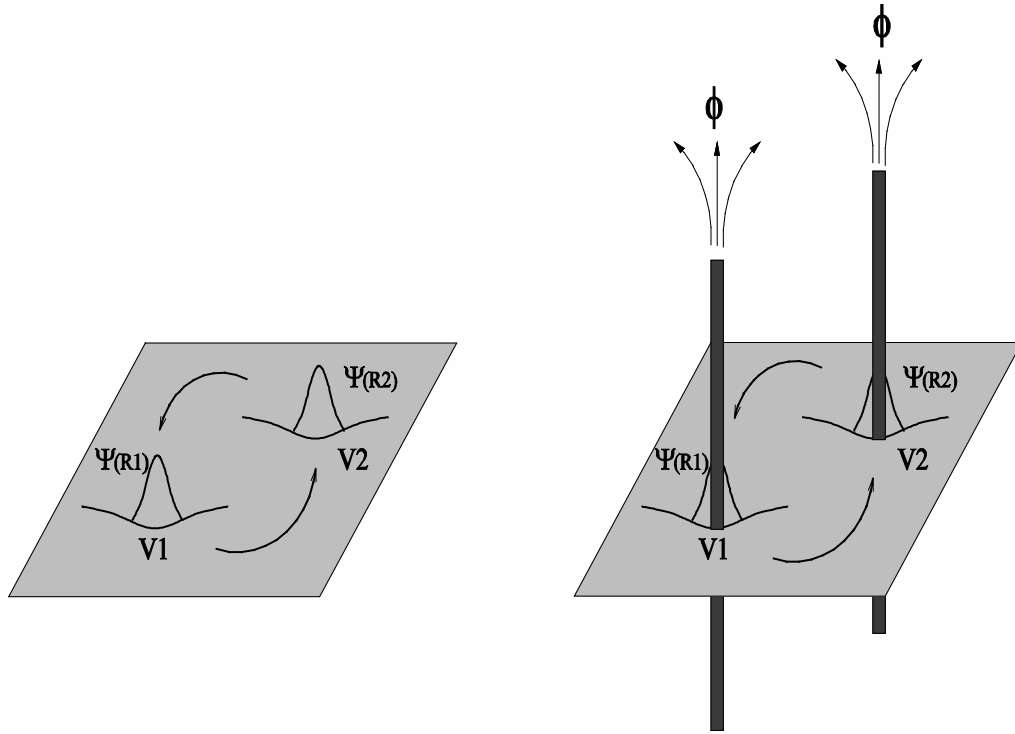


Figure 5.3: Left figure: two-dimensional two-electron system. The electron is localized separately potential valleys $V1$ and $V2$. The valleys are separated such that there is negligible overlap between the electronic wavefunctions. Right figure: the same situation as on the left except now the electrons are independently attached to magnetic flux ϕ and the flux moves along with the electrons.

anyons, the particles with statistics intermediate between Fermi and Bose statistics.

5.4 Anyons

One can change the statistics of the particles in a two-dimensional system based on the fact that the AB phase arises in the presence of magnetic flux. For example, consider a two-electron system, in which each of the electrons are localized in separate potential valleys at \mathbf{R}_1 and \mathbf{R}_2 as shown in Fig. 5.3. We assume that the valleys are separated so that the overlap of the wavefunctions is negligible. The two electron wave

function can then be written in terms of their bound state wavefunctions $\Psi_{\mathbf{R}}(\mathbf{r})$

$$\Psi_{\mathbf{R}_1, \mathbf{R}_2}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}} [\Psi_{\mathbf{R}_1}(\mathbf{r}_1)\Psi_{\mathbf{R}_2}(\mathbf{r}_2) - \Psi_{\mathbf{R}_2}(\mathbf{r}_1)\Psi_{\mathbf{R}_1}(\mathbf{r}_2)] \quad (5.6)$$

Now if we exchange the position of the potential wells adiabatically, the Hamiltonian returns to its original form, but due to the Fermi statistics of the electrons, the total wave function acquires an extra minus sign. Next we consider the situation in which each of the electrons is attached to an infinitesimally thin magnetic flux ϕ as shown in the right figure of Fig. 5.3 and also assume that the flux moves along with the electrons. Again we exchange the position of the potential wells as before, but now there is an extra contribution $e\phi/(2\hbar)$ to the phase due to the flux attachment. We can choose the flux to be such that the additional phase cancels the minus sign due to Fermi statistics. Thus the electrons behave like bosons when a certain amount of flux is attached to them. Since the choice of ϕ is arbitrary, it is possible to make particles with statistics intermediate between that of Fermi and Bose or anyons.

5.5 Composite particle mean field theory

Having introduced the key concepts behind exchange statistics in two-dimensions, we now show how these ideas can be combined into building a field theory applicable in the strongly correlated regime of the rotating Bose gas described by the bosonic-Laughlin wavefunction. We start by defining a unit quantum of flux by $\phi_0 = h/|e|$. Then bosons and fermions can be transformed into fermions and bosons respectively by attaching an odd number of flux quanta to each particle. On the other hand the statistics remains unchanged by attaching an even number of flux quanta. This is shown in Fig. 5.4. Thus electrons in a $\frac{1}{q}$; $q = \text{odd}$ FQHE, can be considered as composite bosons with extra $q\phi_0$ fictitious flux in the opposite direction. The mean field of the fictitious flux cancels with the real flux, and the system can be considered as a system of interacting composite bosons in a zero magnetic field. Thus in the mean field picture, the

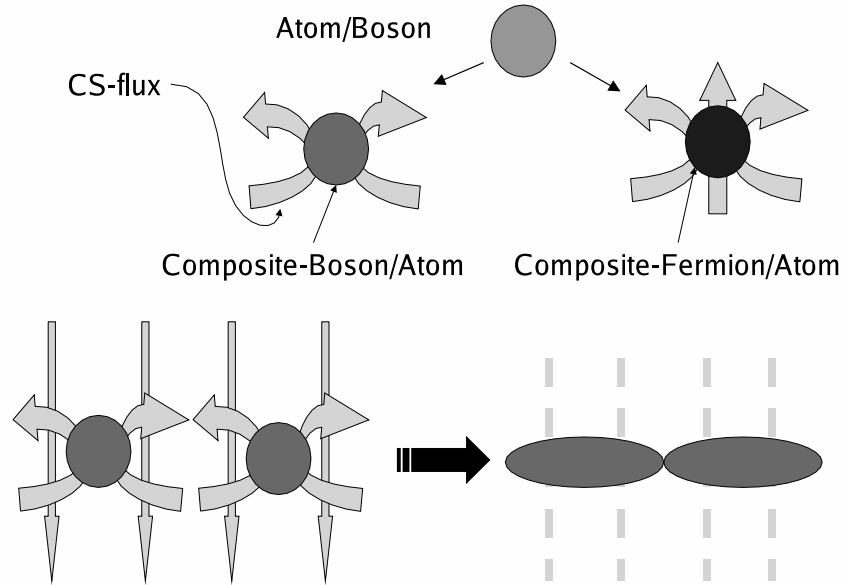


Figure 5.4: Composite particle picture. A boson is transformed to a composite boson(fermion) by attaching an even(odd) number of magnetic flux quanta. The extra fictitious flux cancels the real flux in the mean field approximation resulting in free composite bosons/fermions.

interaction experienced by the composite bosons is just the repulsive coulomb interaction. At the same time we know that a repulsive bosonic system in two dimensions undergoes Bose condensation at zero temperature. Hence in this picture, an odd fractional quantum Hall state can be thought of as a Bose condensed phase of composite bosons.

The same argument holds for our case of bosonic $\frac{1}{2}$ FQHE in rotating Bose gases. Since $q = 2 = \text{even}$, the rotating Bose system can be reduced to a system of interacting composite bosons with zero rotational frequency. The composite bosons still experience the repulsive two-body potential; therefore, the strongly correlated bosonic Laughlin state can be considered as a Bose condensed state of composite bosons.

5.6 Chern-Simons Ginzburg-Landau theory

The starting point of this theory is the second quantized Hamiltonian for atoms(bosons) in the rotating frame. Since we have already made the mapping between the vorticity and magnetic field, we write the second quantized Hamiltonian in terms of the vector potential \mathbf{A} instead of rotating frequency Ω . Also here we specialize to the case of $\Omega = \omega$, so that $H_L = 0$ in (4.3). Thus

$$\hat{H} = \int \hat{\psi}^\dagger(\mathbf{r}) \left[\frac{1}{2m} (-i\mathbf{D})^2 \right] \hat{\psi}(\mathbf{r}) d^2r + \frac{1}{2} \int \int \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}') V(\mathbf{r} - \mathbf{r}') \hat{\psi}(\mathbf{r}) \hat{\psi}(\mathbf{r}') d^2r d^2r' \quad (5.7)$$

where $D_\mu = \partial_\mu + ieA_\mu$ and $\hat{\psi}(\mathbf{r}) = \sum_{l=0}^{\infty} a_l \psi_l^{LL}(\mathbf{r})$ where a_l is the annihilation operator for a particle in the l th single particle lowest landau state. As discussed in the previous section, we would like to work in a composite boson picture. The composite boson field operator $\hat{\phi}$ is defined by (see App. C)

$$\hat{\phi}(\mathbf{r}) = e^{-J(\mathbf{r})} \hat{\psi}(\mathbf{r}) \quad (5.8)$$

where the operator J is given by

$$J(\mathbf{r}) = iq \int d^2r' \rho_a(\mathbf{r}') \theta(z - z') \quad (5.9)$$

In the composite picture the second quantized Hamiltonian retains the same form as given above with the operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ replaced by the corresponding composite boson operators $\hat{\phi}$ and $\hat{\phi}^\dagger$ and D replaced by \mathcal{D} ,

$$\mathcal{D}_\mu = D_\mu + ie a_\mu(\mathbf{r}) \quad (5.10)$$

where a_μ is the fictitious Chern-Simons gauge field. The Chern-Simons field is an auxiliary field subject to certain constraints rather than a dynamical field obeying Maxwell's equation. In order to derive the constraint equation and the stationary solution it is convenient to use the Lagrangian formulation. In the semi-classical approximation,

the Lagrangian density corresponding to the full second quantized Hamiltonian written above is given by

$$\mathcal{L} = \mathcal{L}_a + \mathcal{L}_{cs} \quad (5.11)$$

where

$$\begin{aligned} \mathcal{L}_a &= \varphi^*(\mathbf{r})(i\mathcal{D}_0)\varphi(\mathbf{r}) + \frac{1}{2m}\varphi^*(\mathbf{r})\mathcal{D}^2\varphi(\mathbf{r}) \\ &\quad - \frac{1}{2}\int\varphi^*(\mathbf{r})\varphi^*(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\varphi(\mathbf{r})\varphi(\mathbf{r}')d^2r' \end{aligned} \quad (5.12)$$

$$\mathcal{L}_{cs} = -\frac{e^2}{4q\pi}\varepsilon^{\mu\nu\lambda}a_\mu(\mathbf{r})\partial_\nu a_\lambda(\mathbf{r}) \quad (5.13)$$

This Lagrangian has the usual form except for the last term called the Chern-Simons term. Here μ is the chemical potential and $\varepsilon^{\mu\nu\lambda}$ is the complete antisymmetric tensor with the convention $\varepsilon^{012} = 1$. Therefore the action is given by

$$S = \int\mathcal{L}d^2r dt \quad (5.14)$$

In the rest of the thesis we consider the interaction potentials to be local $V(\mathbf{r}-\mathbf{r}') = V\delta(\mathbf{r}-\mathbf{r}')$. The ground state is the state that minimizes the action S . This implies that the action be invariant under small fluctuations of φ . But first we consider the variation of the action with respect to the zeroth component of the gauge field, $\partial S/\partial a_0 = 0$, which results in the constraint equation

$$\nabla \times \mathbf{a}|_z = -\frac{2q\pi}{e}(\varphi^*\varphi), \quad (5.15)$$

which is essentially the Chern-Simons condition. The extremum condition $\partial S/\partial\varphi^* = 0$ results in the following classical equation of motion for the field φ

$$(i\mathcal{D}_0 + \mu)\varphi(\mathbf{r}) + \frac{1}{2m}\mathcal{D}^2\varphi(\mathbf{r}) - \varphi^*(\mathbf{r})V\varphi(\mathbf{r})\varphi(\mathbf{r}) = 0. \quad (5.16)$$

The most simple solution to the above equation is the one where the Bose field is uniform in space and time $\varphi = \sqrt{\rho}$. Also since the system is homogeneous, to avoid infinite mean

field contribution we shift the zero of energy to this mean field energy. Now we see that the uniform solution is possible only if

$$\mathbf{A} + \mathbf{a} = 0 \quad \text{and} \quad a_0 = 0 \quad (5.17)$$

This implies that $B = (2q\pi/e)\bar{\rho}$ or the filling fraction is $\nu = 1/q$. Thus a uniform solution can exist only when a Laughlin wavefunction can be written down. Thus a bosonic Laughlin state of a rotating Bose gas can be thought of as the Bose condensed phase of composite bosons formed from atoms attached to an even number of vortices.

5.7 Feshbach resonance

From the discussion of the Chern-Simons field theory of the previous section, it is clear that, within this picture, Feshbach interaction can be systematically introduced to tune the inter-atomic interaction. In a typical Feshbach resonance illustrated in Fig. 5.5, the properties of the collision of two atoms are controlled through their resonant coupling to a bound state in a closed channel Born-Oppenheimer potential. By adjusting an external magnetic field, the scattering length can be tuned to have any value. The field dependence of the scattering length is characterized by the detuning ν and obeys a dispersive profile given by

$$a(\nu) = a_{\text{bg}} \left(1 - \frac{\kappa}{2\nu} \right), \quad (5.18)$$

where κ is the resonance width, and a_{bg} is the background scattering length. In fact, all the scattering properties of a Feshbach resonance system are completely characterized by just three parameters

$$U_0 = \frac{4\pi\hbar^2 a_{\text{bg}}}{m}, \quad g = \sqrt{\kappa U_0}, \quad \text{and} \quad \nu. \quad (5.19)$$

Physically, U_0 represents the energy shift per unit density on the single particle eigenvalues due to the background scattering processes, while g , which has dimensions of energy per square-root density, represents the coupling to the Feshbach resonance connecting the open and closed channel potentials.

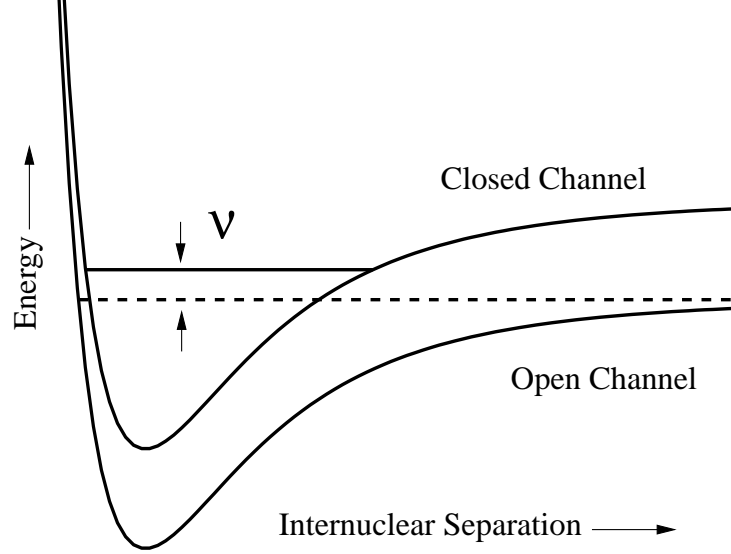


Figure 5.5: A Feshbach resonance results when a closed channel potential possesses a bound state in proximity to the scattering energy in an open channel potential. The detuning of the bound state from the edge of the collision continuum is denoted by ν_0 .

The second quantized Hamiltonian for a dilute Bose gas with binary interactions is given by

$$H = \int d^3x \psi_a^\dagger(\mathbf{x}) H_a(\mathbf{x}) \psi_a(\mathbf{x}) + \int d^3x d^3x' \psi_a^\dagger(\mathbf{x}) \psi_a^\dagger(\mathbf{x}') U(\mathbf{x}, \mathbf{x}') \psi_a(\mathbf{x}') \psi_a(\mathbf{x}) \quad (5.20)$$

where $H_a(\mathbf{x})$ is the single particle Hamiltonian, $U(\mathbf{x}, \mathbf{x}')$ is the binary interaction potential, and $\psi_a(\mathbf{x})$ is the atomic field operator. In cold quantum gases, where the atoms collide at very low energy, we are only interested in the behavior of the scattering about a small energy range above zero. There exist many potentials which replicate the low energy scattering behavior of the true potential; therefore, it is convenient to carry out the calculation with the simplest one, the most convenient choice being to take the interaction potential as a delta-function pseudo potential when possible.

For a Feshbach resonance this choice of pseudo potential is generally not available since the energy dependence of the scattering implies that a minimal treatment must at least contain a spread of wave-numbers which is equivalent to the requirement of a

nonlocal potential. Since the solution of a nonlocal field theory is inconvenient, we take an alternative but equivalent approach. We include in the theory an auxiliary molecular field operator $\psi_m(\mathbf{x})$ which obeys Bose statistics and describes the collision between atoms in terms of two elementary components: the background collisions between atoms in the absence of the resonance interactions and the conversion of atom pairs into molecular states. This allows us to construct a local field theory with the property that when the auxiliary field is integrated out, an effective Hamiltonian of the form given in Eq. (5.20) is recovered with a potential $U(\mathbf{x}, \mathbf{x}') = U(|\mathbf{x} - \mathbf{x}'|)$ which generates the form of the two-body T -matrix predicted by Feshbach resonance theory [104]. The local Hamiltonian which generates this scattering behavior:

$$\begin{aligned}
H = & \int d^3x \psi_a^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m} \nabla_x^2 + V_a(\mathbf{x}) - \mu_a \right) \psi_a(\mathbf{x}) \\
& + \int d^3x \psi_m^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{4m} \nabla_x^2 + V_m(\mathbf{x}) - \mu_m \right) \psi_m(\mathbf{x}) \\
& + \frac{U}{2} \int d^3x \psi_a^\dagger(\mathbf{x}) \psi_a^\dagger(\mathbf{x}) \psi_a(\mathbf{x}) \psi_a(\mathbf{x}) \\
& + \frac{g}{2} \int d^3x \psi_m^\dagger(\mathbf{x}) \psi_a(\mathbf{x}) \psi_a(\mathbf{x})
\end{aligned} \tag{5.21}$$

has the intuitive structure of resonant atom-molecule coupling. Here $V_{a,m}$ are the external potentials and $\mu_{a,m}$ are the chemical potentials. The subscripts a, m represent the atomic and molecular contributions, respectively. The Feshbach resonance is controlled by the magnetic field which is incorporated into the theory by the detuning $\nu = \mu_m - 2\mu_a$ between the atomic and molecular fields. The Hamiltonian in equation (5.21) contains the three parameters U_0 , g , and ν which account for the complete scattering properties of the Feshbach resonance.

Thus in order to introduce Feshbach interaction within the framework discussed earlier in this chapter, we will have to define composite bosons for the molecules exactly in the same way as we did for the atoms, except that the composite molecular bosons will be attached to twice the number of flux quanta as that for the atoms. All these ideas will be put to work in the next chapter.