Chapter 4

Bosonic fractional quantum Hall effect

4.1 Introduction

With the advancements in methods of trapping and cooling atomic gases, a whole new frontier has opened up for the study of many body physics. Due to the extreme control, purity and detailed theoretical understanding of interactions of dilute atomic systems, these systems provide a perfect scenario to test many of the ideas previously limited to condensed matter systems. Since the achievement of Bose-Einstein condensation in weakly interacting atomic gases [1, 2, 3], innumerable papers have been published on the equilibrium properties and dynamical behavior of dilute Bose gases at temperatures above and below the Bose-Einstein condensation temperature, T_c. While many new phenomena have been studied in the regime of weak interactions, the other limit in which the atoms can be strongly correlated has remained unexplored. Strong correlations between particles have shown to produce fascinating effects like the fractional quantum Hall effect (FQHE) in condensed matter systems.

The regime of strong correlations in dilute atomic gases can be achieved by two ways. One possibility is to tune the interactions such that the interaction energy is greater that any other (single particle) energy scale of the problem, for example by using Feshbach resonance [88]. The other possibility is to make the interactions dominate by creating degeneracies. This could be achieved for example by using a periodic trapping potential. Under the conditions that the tunneling amplitude between wells is small compared to the typical interaction energy, the system has been predicted to undergo a transition from a superfluid to a strongly correlated Mott state [89]. Another way of achieving degeneracies is to rotate the harmonic trap that confines the dilute Bose gas. At rotational frequencies close to the trapping frequency, the system is predicted to enter the fractional quantum Hall regime where the ground state is given by the variational Laughlin wavefunction [90]. This is precisely what we will focus in the remaining of this thesis.

In this chapter, we will discuss the fractional quantum Hall effect (FQHE) in brief. Then we will see how a dilute Bose gas can be prepared under conditions identical to those required for observing the electronic FQHE. Having met the requirements, with simple quantum mechanical arguments we will show that this bosonic system exhibits single and many-particle states identical to that of the electronic FQHE system.

4.2 Fractional quantum Hall effect

The quantization of the Hall effect discovered by von Klitzing, Dorda, and Pepper [91] in 1980 is a remarkable macroscopic quantum phenomenon which occurs in two-dimensional electron systems at low temperature and strong perpendicular magnetic field. Under these conditions the Hall-conductivity exhibits plateaus at integral multiples of a fundamental constant of nature e^2/h. This is the integer quantum Hall effect (IQHE). The striking result is the accuracy of the quantization irrespective of the impurities or the geometric details of the two-dimensional system. Each plateau is associated with a deep minimum in the diagonal resistivity indicating a dissipationless flow of current. This was followed by the discovery of fractional quantum Hall effect (FQHE) by Tsui, Stormer, and Gossard [92] in higher mobility samples where the plateaus occur at fractional multiples of e^2/h as shown in Fig. 4.1. Despite the apparent similarity of experimental results, the physical mechanism responsible for the IQHE and the FQHE are quite different. In the former the random impurity potential plays an important role

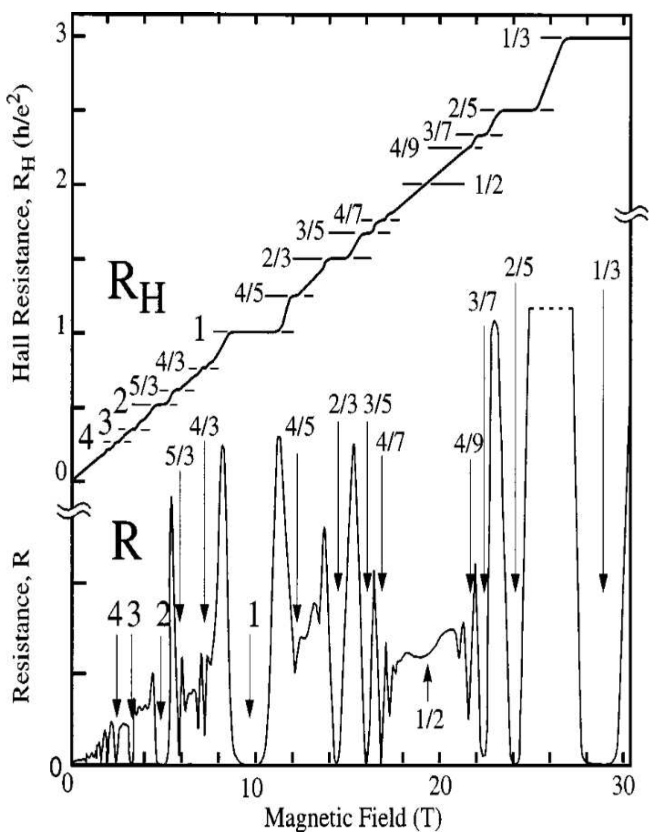

Figure 4.1: The FQHE in ultra high-mobility modulation-doped GaAs/AlGaAs. Many fractions are visible.

while in the latter it is the collective effect arising from the coulombic electron-electron repulsion. Also note that the fractional values originally observed as shown in Fig. 4.1 were all odd denominator fractions. While this is in fact related to the fermionic nature of electrons, more recently [93, 94] even fractions like $\frac{5}{2}$ have also been observed. Occurance of such even fractions is rare, and their existence can be attributed to a Cooper instability [95]; nevertheless, it is again a collective phenomenon due to the interaction between electrons.

Regardless of the nature of the fractions, the many body physics that is important to the current discussion is that the fractional quantum Hall state is a liquid state exhibiting a gap in the excitation spectrum. This has been very well understood by methods based on exact diagonalization and trial wavefunctions. The variational ground state proposed by Laughlin [90] has been very successful in explaining the FQHE and predicts the existence of quasi-particles with fractional charge. In the forthcoming sections we show that identical many body effect can be predicted for a rotating Bose gas. In fact a bosonic variant of the many body Laughlin wavefunction can be written down to describe the ground state of the system in this strongly correlated regime.

4.3 Electron system versus dilute Bose gas

While at first glance a rotating Bose gas may seem very different from the fractional quantum Hall system, there exists an interesting symmetry with respect to the measurable physical quantities between these systems. This is summarized in the table below where we list the necessary conditions and physical quantities for a rotating Bose gas corresponding to those which define the electronic fractional quantum Hall system.

	Electronic FQHE system	**Rotating Bose gas**
1	2-dimensional at interface between semiconductors	effective 2-dimensional due to strong confinement along the z direction compared to the x and y directions
2	Repulsive coulombic interaction	Repulsive two body potential
3	Vector potential $\mathbf{A}$	Velocity field $\mathbf{u}$
4	Magnetic field $\mathbf{B}$ and magnetic flux ϕ, $\mathbf{B} = \nabla \times \mathbf{A}$; $\phi = \int \mathbf{B} \cdot \hat{n} dS$	Vorticity $\mathbf{\Omega}$ and circulation Θ, $\mathbf{\Omega} = \nabla \times \mathbf{u}$; $\Theta = \int \mathbf{\Omega} \cdot \hat{n} dS$
5	Landau levels fermionic-Laughlin state	Landau levels bosonic-Laughlin state

Table 4.1: Comparison between the electronic fractional quantum Hall system and the rotating Bose gas.

As mentioned previously, the FQHE occurs in a two-dimensional electron system in the presence of a strong perpendicular magnetic field. Identical conditions can be achieved for a dilute Bose gas by having the confinement in the z direction strong compared to the x and y directions while the vorticity $\mathbf{\Omega}$ of the rotating Bose gas plays the role of the $\mathbf{B}$ field. The many body effects are generated by the repulsive two-body inter-atomic potential which replaces the coulombic repulsion for electrons. Thus, apart from the intrinsic statistics and charge of the particles, it is theoretically possible to configure a dilute Bose gas to meet the requirements necessary for observing a FQHE. Therefore in principle one could predict a strongly correlated ground state for the rotating Bose gas.

To begin with we will first address the last row of the above table by showing that the Landau levels describe the single particle spectrum of a rotating Bose gas in complete analogy to the electronic case. By including inter-atomic repulsive interactions we will also show that a bosonic variant of the Laughlin wavefunction can be written down as the variational many body ground state of the rotating Bose gas.

4.4 Landau levels and the many-body Laughlin wavefunction

We start with writing the Hamiltonian for N interacting bosonic atoms confined to a harmonic potential in the rotating frame. Also, we assume the confinement along the axis of rotation to be strong enough that we can neglect the excitations in that direction and consider the system to be essentially two dimensional:

$$H = \sum_{j=1}^{N} \left\{ \frac{p_{jx}^2}{2m} + \frac{p_{jy}^2}{2m} + \frac{m\omega^2 x_j^2}{2} + \frac{m\omega^2 y_j^2}{2} - \Omega L_{jz} \right\} + \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j). \tag{4.1}$$

Here, m is the mass of the atoms, the angular momentum operator is given by $L_z = xp_y - p_x y$, and Ω is the rotating frequency. The repulsive two body potential V depends only on the relative coordinate of the individual atoms. Now if we make the following identification

$$A_{jx} = \frac{m\omega}{e} y_j; \quad A_{jy} = -\frac{m\omega}{e} x_j, \tag{4.2}$$

the above Hamiltonian can be cast in a form very similar to the fractional quantum Hall Hamiltonian

$$H = \sum_{j=1}^{N} \left\{ \frac{1}{2m}(p_{jx} + eA_{jx})^2 + \frac{1}{2m}(p_{jy} + eA_{jy})^2 \right\} + (\omega - \Omega) \sum_{j=1}^{N} L_{jz} + \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j). \tag{4.3}$$

To obtain the single particle spectrum we will first consider the non-interacting case, i.e. $V = 0$. The Hamiltonian can then be rewritten in a more convenient form

$$H = \sum_{j=1}^{N} \left(a_j^\dagger a_j + \frac{1}{2} \right) 2\hbar\omega + \hbar(\omega - \Omega) \sum_{j=1}^{N} (b_j^\dagger b_j - a_j^\dagger a_j), \tag{4.4}$$

where the operators a and b are defined in terms of the covariant momentum $(P_x, P_y) = (p_x + eA_x, p_y + eA_y)$ and the guiding center coordinate $(X, Y) = (x + \frac{1}{2m\omega} P_y, y - \frac{1}{2m\omega} P_x)$

$$a_j = \sqrt{\frac{1}{4\hbar m\omega}} (P_{jx} + iP_{jy}); \quad a_j^\dagger = \sqrt{\frac{1}{4\hbar m\omega}} (P_{jx} - iP_{jy}) \tag{4.5}$$

and

$$b_j = \sqrt{\frac{\hbar}{4m\omega}} (X_j - iY_j); \quad b_j^\dagger = \sqrt{\frac{\hbar}{4m\omega}} (X_j + iY_i) \tag{4.6}$$

obeying the following commutation relations

$$[a_i, a_j^\dagger] = [b_i, b_j^\dagger] = \delta_{ij}; \tag{4.7}$$

$$[a_i, a_j] = [a_i, b_j] = [a_i^\dagger, b_j] = [b_i, b_j] = 0. \tag{4.8}$$

The operators a and $a^\dagger$ annihilate and create a quanta in the Fock space, while the operators b and $b^\dagger$ are the usual angular momentum ladder operators. Now one can easily see that in the absence of interactions, at a certain critical rotational(stirring) frequency $\Omega = \omega$, the Hamiltonian depends only on the a's and $a^\dagger$'s. The corresponding single particle energy states are the Landau levels given by

$$|n_j, l_j\rangle = \sqrt{\frac{1}{n_j!(l_j + n_j)!}} (a_j^\dagger)^{n_j} (b_j^\dagger)^{l_j + n_j} |0\rangle \tag{4.9}$$

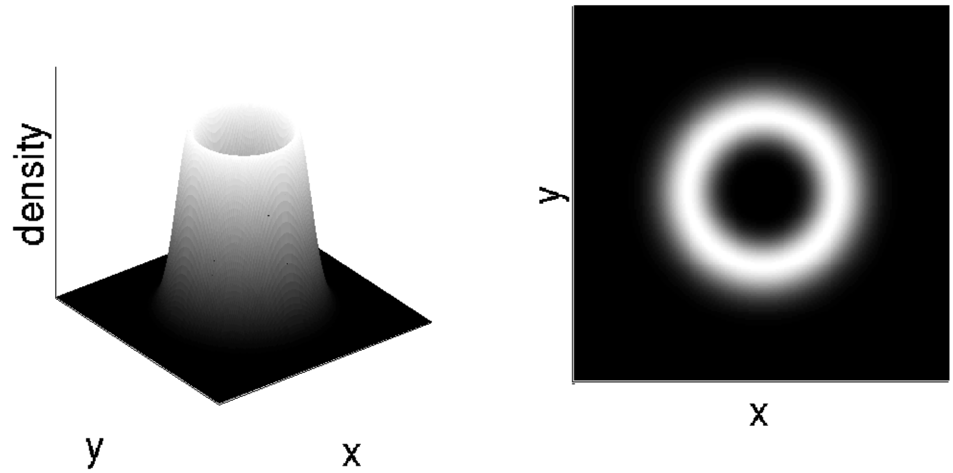

Figure 4.2: Atomic density in the lowest Landau level: left - density in the lowest Landau level with angular momentum quantum number $l = 3$ is sharply peaked at radius $r = \sqrt{l\hbar/m\omega}$, right - density is shown by a concentric ring of thickness $\Delta r = \sqrt{\hbar/4lm\omega}$ in the x-y plane.

with energy eigenvalues $E_{n_j} = (n_j + \frac{1}{2})2\hbar\omega$ and angular momentum $L_j = l_j\hbar$, where $|0\rangle$ is the Fock vacuum. Since the angular momentum operator commutes with the Hamiltonian, each Landau level is degenerate with respect to the angular momentum quantum number l. Now using the action of the operators a and b on the Fock states, the configuration space wavefunction of the zero angular momentum lowest Landau state can be easily shown to be given by

$$\psi_0^0(\mathbf{x}) = \sqrt{\frac{m\omega}{\pi\hbar}} \exp\left(-\frac{m\omega|\mathbf{x}|^2}{2\hbar}\right). \tag{4.10}$$

Therefore using Eq.(4.9), the generic Landau wavefunction can then be written as

$$\psi_l^n(\mathbf{x}) = \langle \mathbf{x}|n, l\rangle = \sqrt{\frac{1}{n!(l+n)!}}(a^\dagger)^n(b^\dagger)^{l+n}\psi_0^0(\mathbf{x}). \tag{4.11}$$

The atomic density in the lowest Landau level given by $|\psi_l^0(\mathbf{x})|^2$ is sharply peaked at radius $r = \sqrt{l\hbar/m\omega}$ as depicted in Fig. 4.2. The lowest Landau states are therefore represented by concentric rings labeled by the angular momentum quantum number l (Fig. 4.2). Thus the single particle spectrum for the rotating Bose gas shows Landau level structure identical to that of the electrons in a strong perpenticular magnetic field.

Now we include interactions in order to calculate the many-body ground state. We start by writing the Hamiltonian (4.3) as

$$H = H_{\text{Landau}} + H_L + V \tag{4.12}$$

Here H_{Landau} is the quantum Hall single particle Hamiltonian whose single particle states are the Landau levels separated by $2\hbar\omega$. The Hamiltonian $H_L = (\omega - \Omega)L_z$ is proportional to the z component of the total angular momentum $L_z = \sum_{j=1}^{N} L_{jz}$, and V is the interaction term. From now on we would like to consider the limit in which the energy scale characterizing the Hamiltonian H_{Landau} and V is much greater than the one corresponding to H_L. This is possible when the stirring frequency is very close to the harmonic trapping frequency. Also we will consider only contact interaction so that V can be written as

$$V = \eta \sum_{i<j}^{N} \delta(\mathbf{x}_i - \mathbf{x}_j). \tag{4.13}$$

In this limit, the ground states and elementary excitations of the system will lie in the subspace of common zero energy eigenstates of H_{Landau} and V. Thus the many body wavefunction $\Psi(z)$ must lie within the subspace generated by the tensor product of the lowest Landau level single particle states

$$\Psi(\{z\}) = \mathcal{P}(z_1, ..., z_N) \prod_k \exp\left(-\frac{|z_k|^2}{2}\right), \tag{4.14}$$

where $\mathcal{P}(z)$ is a polynomial in each of the scaled complex atomic coordinates

$$z_j = \sqrt{\frac{m\omega}{\hbar}}(x_j + iy_j). \tag{4.15}$$

Now if $\Psi(z)$ is also an eigenstate of V, then due to the delta function nature of V, the polynomial $\mathcal{P}(z)$ has to satisfy the form

$$\mathcal{P}(\{z\}) = \mathcal{Q}(\{z\}) \prod_{i<j}(z_i - z_j)^2 \tag{4.16}$$

Here, the "2" in the exponent, is due to the even symmetry of bosons under exchange. We can now diagonalize H_L within the truncated Hilbert space of wavefunctions of the form specified by (4.14) and (4.16). At the same time when $\mathcal{P}(z)$ is a homogeneous polynomial in z, the state (4.14) is an eigenstate of H_L with eigenvalue $E_M = \hbar(\omega - \Omega)M$, where M is the homogeneous degree of $\mathcal{P}(z)$. Now since the ground state is the state with the lowest angular momentum, we conclude that $\mathcal{Q}(z) = 1$ and the ground state wavefunction can be written as

$$\Psi(\{z\}) = \prod_{i<j}(z_i - z_j)^2 \prod_k \exp\left(-\frac{|z_k|^2}{2}\right). \tag{4.17}$$

This is the essentially the bosonic variant of the Laughlin wave function [96, 97] originally proposed for fractional quantum Hall electrons. The most generic Laughlin wavefunction is represented by

$$\Psi_q(\{z\}) = \prod_{i<j}(z_i - z_j)^q \prod_k \exp\left(-\frac{|z_k|^2}{2}\right) \tag{4.18}$$

where $\frac{1}{q}$ represents the fraction in the FQHE. Therefore as one can immediately notice that, q has to be odd(even) for the case of electrons(bosons) to satisfy the Fermi(Bose) symmetry. Physically the fraction $\frac{1}{q}$, represents the number of electrons(bosons) per flux-quanta(vortex) and is called the filling fraction. Thus the Laughlin wave function (4.17) for the rotating Bose gas represents a $\frac{1}{2}$ bosonic FQHE. The atomic density profile in the $\frac{1}{2}$-Laughlin wavefunction is shown in Fig. 4.3.

4.5 Exact diagonalization

To confirm the validity of the arguments presented in the previous section we numerically diagonalize the Hamiltonian (4.3) for a small system of five bosonic atoms. As before we assume hard core interaction potential represented by a delta function. Also in order to limit the basis states to the lowest Landau levels, the interaction parameter η is chosen such that the interaction energy is small compared to the Landau level spacing, $2\hbar\omega$. We choose the stirring frequency to be smaller than but very close to

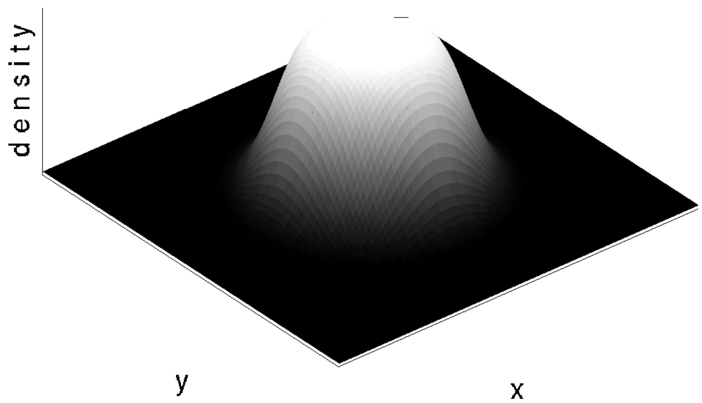

Figure 4.3: The atomic density in the $\frac{1}{2}$- Laughlin state for $N = 5$ atoms.

the trapping frequency, $1 - \Omega/\omega = 0.01$. Such a choice allows us to see the effect of the H_L piece of the Hamiltonian while satisfying the assumptions of the previous section. In Fig. 4.4 we plot the eigenenergies as a function of the total angular momentum L_{tot}. We notice that there is a branch of states well separated from the rest of the spectrum. These states are polynomial states of the form given by (4.14) and (4.16). The ground state is the state with angular momentum $L_{\text{tot}} = N(N-1) = 20$ and is the Laughlin state corresponding to the bosonic $\frac{1}{2}$ FQHE.

4.6 Quasiparticles

The Laughlin wavefunction (4.17) describes a bound state of atoms with even number of zeros or vortices. Now if the area of the system is slightly increased, the number of vortices increases, resulting in a change of the filling fraction from $\frac{1}{2}$. Now consider the case where we introduce one extra vortex. The new wavefunction can then be written as

$$\Psi_{2+\varepsilon}(z) = \prod_i z_j \Psi_2(z) \tag{4.19}$$

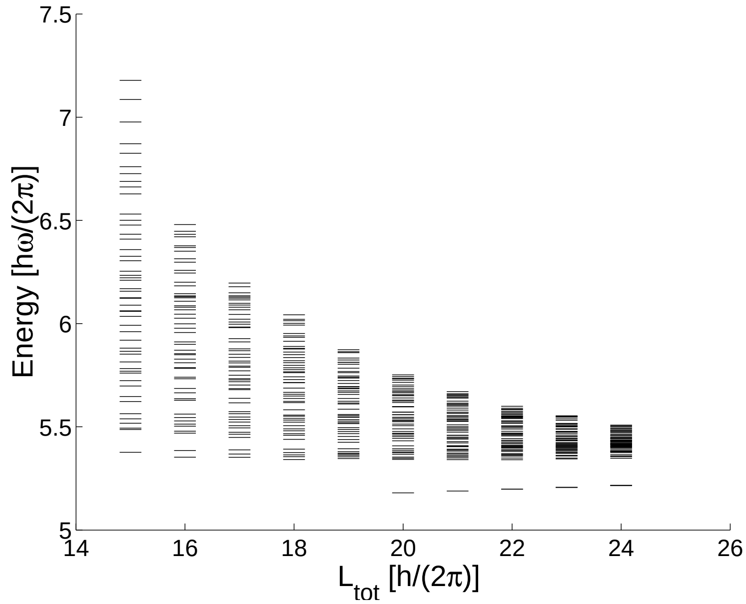

Figure 4.4: Results of exact diagonalization for a small system of $N = 5$ bosonic atoms in a rotating 2-dimensional harmonic trap. The stirring frequency is chosen to be smaller than but very close to the trapping frequency, $1 - \Omega/\omega = 0.01$. The parameter η is chosen such that the interaction energy is very small compared to the Landau level spacing, $2\hbar\omega$. Such a choice allows us to use a truncated Hilbert space of lowest Landau levels.

and possesses a zero at the origin. Physically, the above operation shifts the atomic distribution corresponding to $(l_1, l_2, ..l_N)$ to the one corresponding to $(l_1+1, l_2+1, ..l_N+1)$. This can be verified trivially by expanding the above wavefunction and will be left to the reader. From the experimental point of view, such a situation can be realized by shining a laser at $z = 0$ as proposed by Paredes et al. in [97]. More generally we could shine the laser at some point z_0 (localized within an area $\sim \hbar/(m\omega)$) in which case the Laughlin wavefunction will be multiplied by $\prod_j (z_j - z_0)$ and hence reducing the atomic density at that point to zero. Such an isolated zero acts like a quasi-hole excitation with fractional statistics. This is very easy to see: consider a hypothetical situation in which we introduce two zeros at the same point z_0, and then pin an extra atom to that point. Such an operation would return the ground state to the $\frac{1}{2}$ Laughlin wavefunction of the previous section except now for $(N + 1)$ bosons. Thus two quasiholes are annihilated by an atom and hence the quasi-hole corresponds to a particle with $\frac{1}{2}$ statistics. The rotating Bose gas can also be predicted to be incompressible. This is because compressing or expanding the state would require removal or injection of an atom requiring a finite amount of energy. In essence, it is not possible to induce an infinitesimal change in the system by an infinitesimal change in the pressure.

4.7 Conclusion

In this chapter we showed that the rotating Bose gas under certain conditions represents a strongly correlated system in complete analogy to the electronic FQHE system. Here the ground state is given by the bosonic variational $\frac{1}{2}$- Laughlin wave-function. The $\frac{1}{2}$- quasi-hole excitations can be created by shining a laser at some specific point.

While theoretical understanding of the occurence of such a strongly correlated state is quite clear, experimental realization seems difficult. The main difficulty is due to the extreme low temperature required to access this state. Typically the temperature required is such that [97]

$$\frac{kT}{\hbar\omega} \ll \frac{1}{N}, \frac{\eta}{N}. \tag{4.20}$$

The remainder of this thesis will be devoted to precisely address this issue. We will be considering an alternative approach, where along with creating degeneracies by rotating the trap, we increase the interaction energy by applying Feshbach resonance. The motivation for this approach can be understood from the numerical results of Fig. 4.4 and the nature of the Laughlin wavefunction. The strongly correlated Laughlin state corresponds to the lowest energy state in the separated branch of states. This is because, the Laughlin wavefunction is annihilated by the delta function interaction potential. Therefore, except for the states in this branch, eigenenergies of all the other states have a contribution corresponding to the interaction energy. Therefore, naively, increasing the interaction energy would increase the gap between the separated branch and the continuum resulting in an increased temperature for observing the ground state.

However, in implementing such an approach, an alternative formulation of the rotating Bose gas is required. This is because, while the physics of such resonances is well formulated in a mean field picture, mean field theories fail to describe the strongly correlated regime. The order parameter describing for example the Laughlin state is no longer given by the expectation value of the atomic field operator. This difficulty can be resolved in a gauge transformed Chern-Simons approach [98, 99]. Here the problem can be formulated in terms of mean fields of composite particles. In the next chapter, we discuss the basic ideas behind this effective field theory picture. Then continuing along the same lines, in Ch. 6 we predict novel strongly correlated states in the presence of Feshbach resonance.