Appendix C

Anyon field operator

The anyon field operator $\hat{\phi}_{an}(t,\mathbf{x})$ is represented by

$$\hat{\phi}_{an}(t,\mathbf{x}) = e^{i\alpha\hat{\xi}(t,\mathbf{x})}\hat{\psi}_b(t,\mathbf{x}) \tag{C.1}$$

in terms of the boson operator $\hat{\psi}_b(t,\mathbf{x})$ with the phase field

$$\hat{\xi}(t,\mathbf{x}) = \alpha\int d^2y\theta(\mathbf{x}-\mathbf{y})\rho(t,\mathbf{x}). \tag{C.2}$$

where $\rho(t,\mathbf{x}) = \hat{\psi}_b^\dagger(t,\mathbf{x})\hat{\psi}_b(t,\mathbf{x})$. The Chern-Simons field is expressed as

$$a_k(\mathbf{x}) = \frac{\alpha\hbar}{e}\int d^2y\partial_k\theta(\mathbf{x}-\mathbf{y})\rho(\mathbf{y}). \tag{C.3}$$

We now examine how the exchange phase arises theoretically. We start with the commutation relation

$$[\hat{\xi}(\mathbf{y}),\hat{\psi}_b(\mathbf{x})] = \int d^2z\theta(\mathbf{x}-\mathbf{y})[\rho(\mathbf{z}),\hat{\psi}_b(\mathbf{x})] = -\theta(\mathbf{y}-\mathbf{x})\hat{\psi}_b(\mathbf{x}). \tag{C.4}$$

Now, we use the Hausdorff formula

$$e^ABe^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + ..., \tag{C.5}$$

by setting $A = i\alpha\hat{\xi}(\mathbf{y})$ and $B = \hat{\psi}_b(\mathbf{x})$ we find that

$$e^{-i\alpha\hat{\xi}(\mathbf{y})}\hat{\psi}_b(\mathbf{x})e^{i\alpha\hat{\xi}(\mathbf{y})} = e^{i\alpha\theta(\mathbf{y}-\mathbf{x})}\hat{\psi}_b(\mathbf{x}). \tag{C.6}$$

With this formula we get

$$\hat{\phi}_{an}(\mathbf{x})\hat{\phi}_{an}(\mathbf{y}) = e^{i\alpha\theta(\mathbf{y}-\mathbf{x})}e^{i\alpha\hat{\xi}(\mathbf{x})}e^{i\alpha\hat{\xi}(\mathbf{y})}\hat{\psi}_b(\mathbf{x})\hat{\psi}_b(\mathbf{y}) \tag{C.7}$$

$$\hat{\phi}_{an}(\mathbf{y})\hat{\phi}_{an}(\mathbf{x}) = e^{i\alpha\theta(\mathbf{x}-\mathbf{y})}e^{i\alpha\hat{\xi}(\mathbf{y})}e^{i\alpha\hat{\xi}(\mathbf{x})}\hat{\psi}_b(\mathbf{y})\hat{\psi}_b(\mathbf{x}). \tag{C.8}$$

Now using $\hat{\psi}_b(\mathbf{x})\hat{\psi}_b(\mathbf{y}) = \hat{\psi}_b(\mathbf{y})\hat{\psi}_b(\mathbf{x})$ and $\theta(\mathbf{x}-\mathbf{y}) = \theta(\mathbf{y}-\mathbf{x}) + \pi$, we compare (C.7) and (C.8) to find that

$$\hat{\phi}_{an}(\mathbf{y})\hat{\phi}_{an}(\mathbf{x}) = e^{i\alpha\pi}\hat{\phi}_{an}(\mathbf{x})\hat{\phi}_{an}(\mathbf{y}) \tag{C.9}$$

This is the operator version of the interchange of two anyons.