APPENDIX C

DERIVATION OF INTRACAVITY INTENSITY WITH SIDEBANDS

This appendix derives the intensity inside a Fabry-Perot etalon when the incident light is phase modulated at the free spectral range of the etalon. This intensity is given in Eq. (7.2).

An electric field of the form

$$\epsilon = A\cos(\omega t - \delta \sin \omega_m t) \tag{C.1}$$

can be written in terms of Bessel functions as [74]

$$\begin{aligned} \epsilon = A[J_0(\delta)\cos\omega t \quad &+ \quad 2J_1(\delta)\sin\omega t \sin\omega_m t \\ &+ \quad 2J_2(\delta)\cos\omega t\cos 2\omega_m t + \ldots] \end{aligned} \tag{C.2}$$

where the series extends including higher order Bessel functions and higher multiples of ω_m, and δ is the phase modulation index. If the field is part of an electromagnetic wave, it must have spatial dependence as well. If we replace ωt with $\omega t - ky$ and replace $\omega_m t$ with $\omega_m t - k_m y$, and use the identities

$$\cos\theta\cos\phi = \frac{1}{2}[\cos(\theta-\phi) + \cos(\theta+\phi)] \tag{C.3}$$

$$\sin\theta\sin\phi = \frac{1}{2}[\cos(\theta-\phi) - \cos(\theta+\phi)] \tag{C.4}$$

we get a wave traveling in the $\hat{y}$ direction with the form

$$\begin{aligned}\epsilon = A\{J_0(\delta)\cos(\omega t - ky) &+ J_1(\delta)\left[\cos(\omega_- t - k_- y) - \cos(\omega_+ t - k_+ y)\right] \\ &+ J_2(\delta)\left[\cos(\omega_{--} t - k_{--} y) + \cos(\omega_{++} t - k_{++} y)\right]\}\end{aligned} \tag{C.5}$$

where $k_\pm = k \pm k_m$ and $\omega_\pm = \omega \pm \omega_m$. If this wave is injected into a stable Fabry-Perot etalon with a free spectral range of $\nu_{\mathrm{FSR}} = 2\pi\omega_m$, then the standing wave formed inside the etalon is given by

$$\begin{aligned}\epsilon = 2A[J_0 \sin\omega t \sin ky &+ J_1\left(\sin\omega_- t \sin k_- y - \sin\omega_+ t \sin k_+ y\right) \\ &+ J_2\left(\sin\omega_{++} \sin k_{++} y + \sin\omega_{--} \sin k_{--} y\right)],\end{aligned} \tag{C.6}$$

where the label δ for the phase modulation index has been dropped. Using the identities $\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$, the electric field can be written as

$$\begin{aligned}\epsilon &= 2A[J_0 \sin\omega t \sin ky \\ &- 2J_1\left(\cos\omega t \sin\omega_m t \sin ky \cos k_m y + \sin\omega t \cos\omega_m t \cos ky \sin k_m y\right) \\ &+ 2J_2\left(\cos\omega t \sin 2\omega_m t \cos ky \sin 2k_m y + \sin\omega t \cos 2\omega_m t \sin ky \cos 2k_m y\right)].\end{aligned} \tag{C.7}$$

We square the field to find the intensity, and since the atoms do not react to changes in the intensity on time scales near $2\pi/\omega$, or even $2\pi/\omega_m$, we will take the time average of the intensity. Therefore we will drop terms containing odd trigonometric functions or products of orthogonal trigonometric functions. The time averaged intensity inside the cavity is given by

$$\begin{aligned}\langle\epsilon^2\rangle_t = 2A^2\,[\,J_0^2 \sin^2 ky &+ 2J_1^2\left(\sin^2 ky \cos^2 k_m y + \cos^2 ky \sin^2 k_m y\right) \\ &+ 2J_2^2\left(\sin^2 ky \cos^2 2k_m y + \cos^2 ky \sin^2 2k_m y\right)\,].\end{aligned} \tag{C.8}$$

Using the identities $\cos^2\theta = 1 - \sin^2\theta$ and $\cos^2\theta - \sin^2\theta = \cos 2\theta$, and setting

$A = 1/\sqrt{2}$, the time averaged intensity inside the cavity can finally be written as

$$\begin{aligned} \langle \epsilon^2 \rangle_t = \sin^2 ky[J_0^2(\delta) &+ 2J_1^2(\delta)\cos 2k_m y + 2J_2^2(\delta)\cos 4k_m y] \\ &+ 2[J_1^2(\delta)\sin^2 k_m y + J_2^2(\delta)\sin^2 2k_m y]. \end{aligned} \tag{C.9}$$