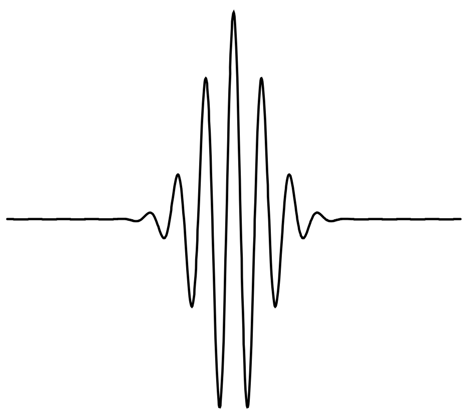
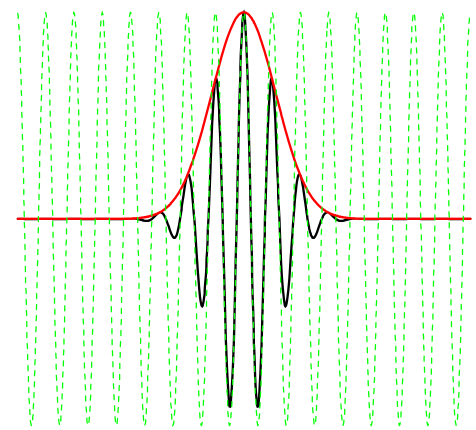
How do we describe a pulse in the time domain?

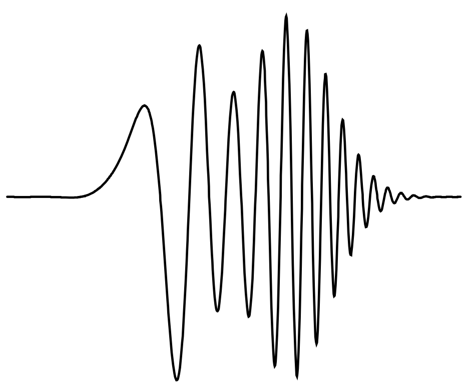
Full & unambiguous:

$E(t)$

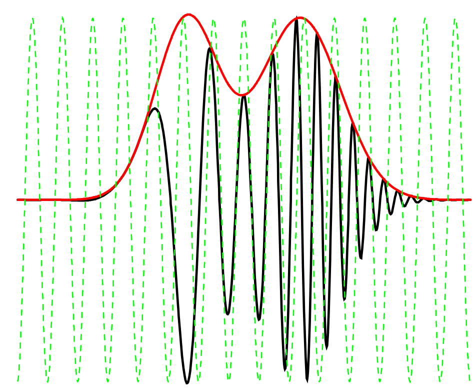
Decomposition into carrier and envelope: $E(t) = \hat{E}(t)\cos(\omega t + \phi)$







?



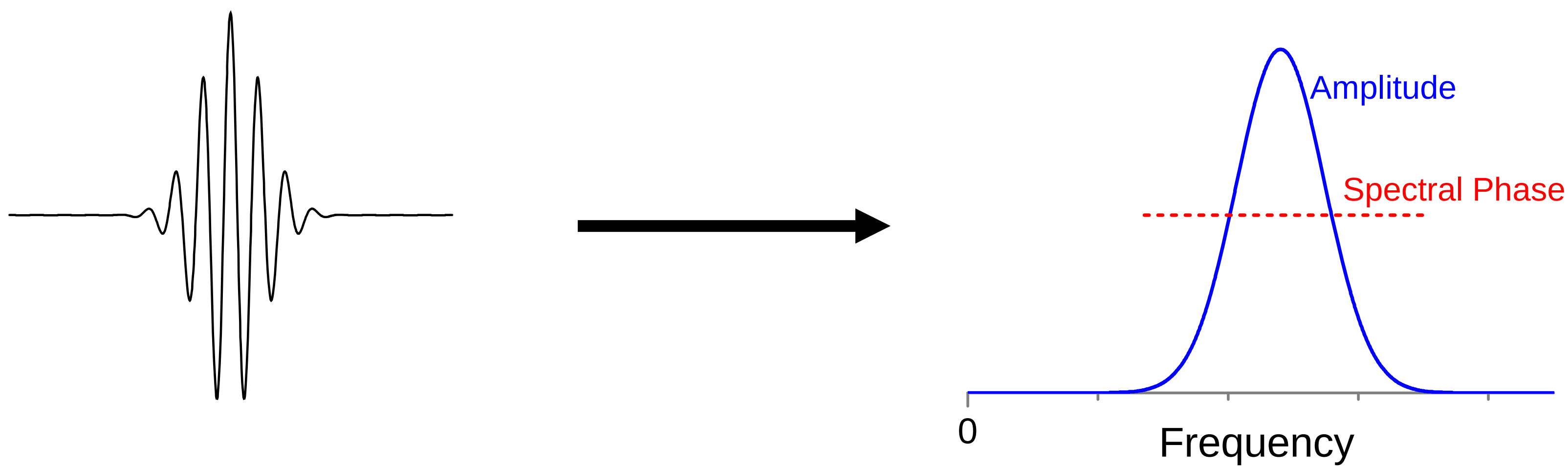
The decomposition is fundamentally arbitrary

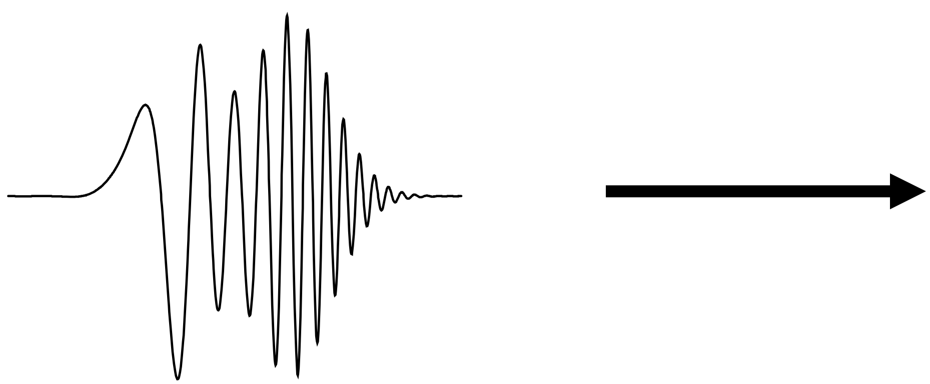
- particularly for shorter pulses
- but useful

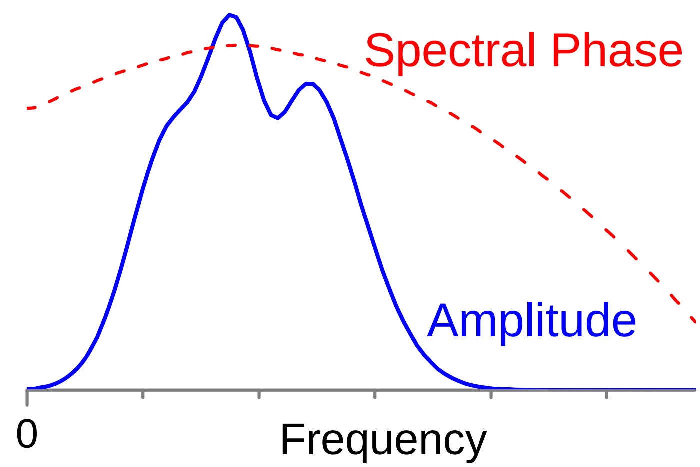
Originally discussed in the context Of radar pulses, called "analytic signal Theory", see Gabor, J. Inst. Elect. Eng. **93** Pt., III 429 (1946) or Mandel and Wolf text.

Frequency domain description

Need field (amplitude spectrum) including phase information



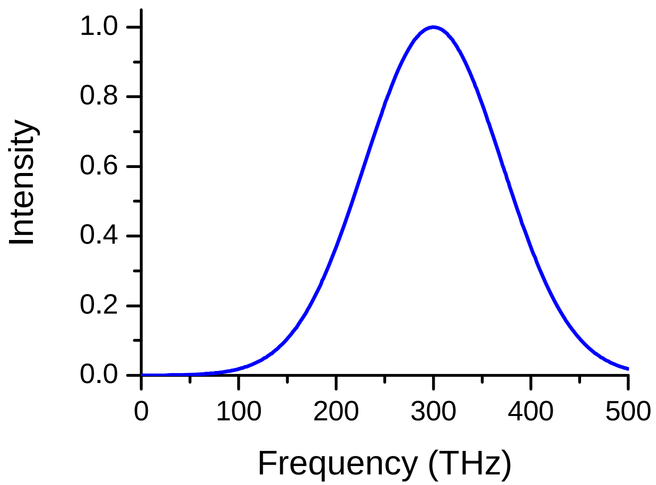


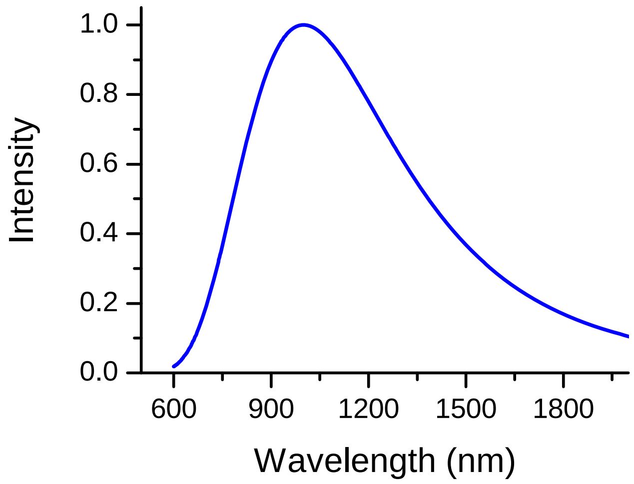


Assumption:
Pulse is centered at $t = 0$
→ Remove linear phase

Frequency vs. wavelength

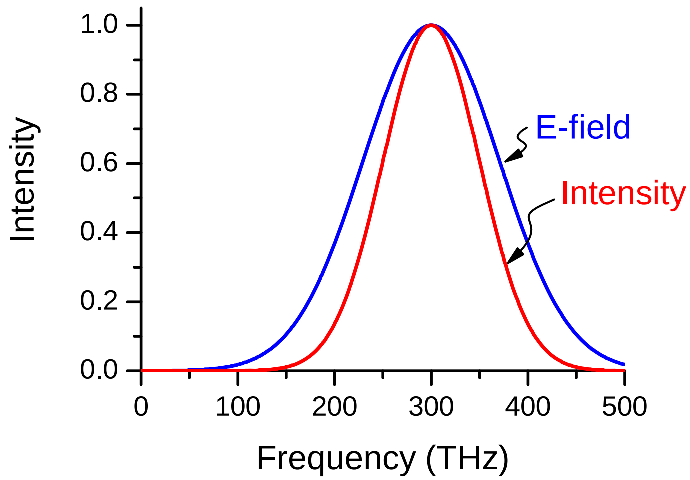
For large bandwidth, the spectrum looks quite different in frequency vs. wavelength





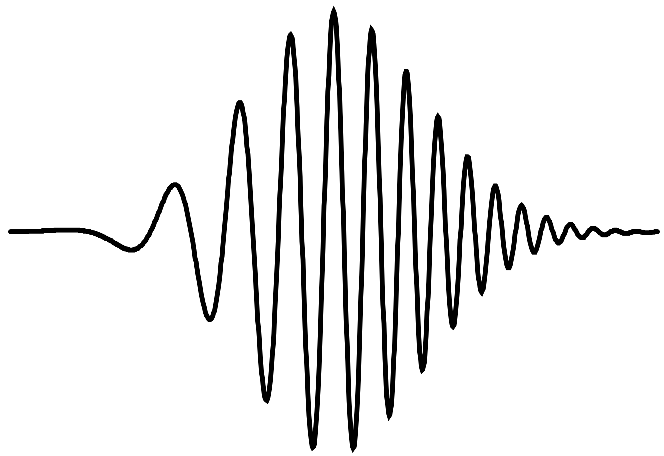
Amplitude vs. Intensity

Similarly, the spectral or temporal intensity envelope is different (narrower) than the e-field envelope

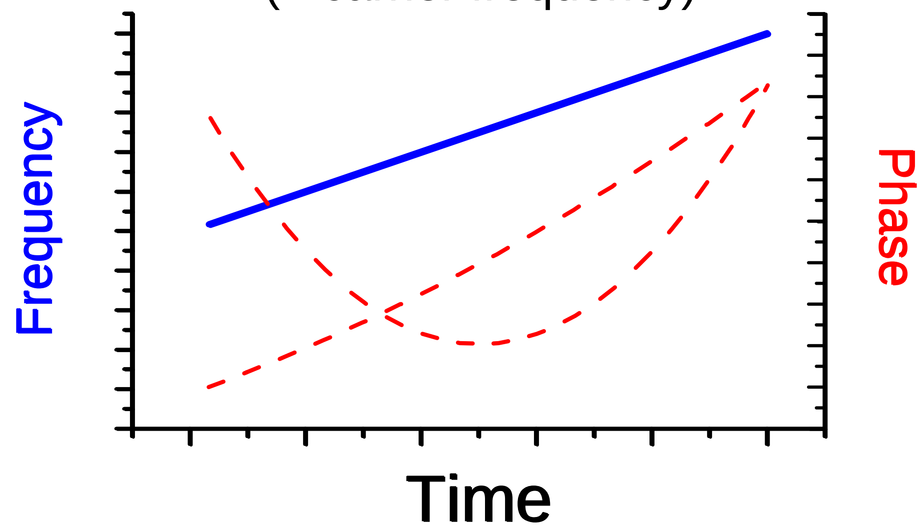


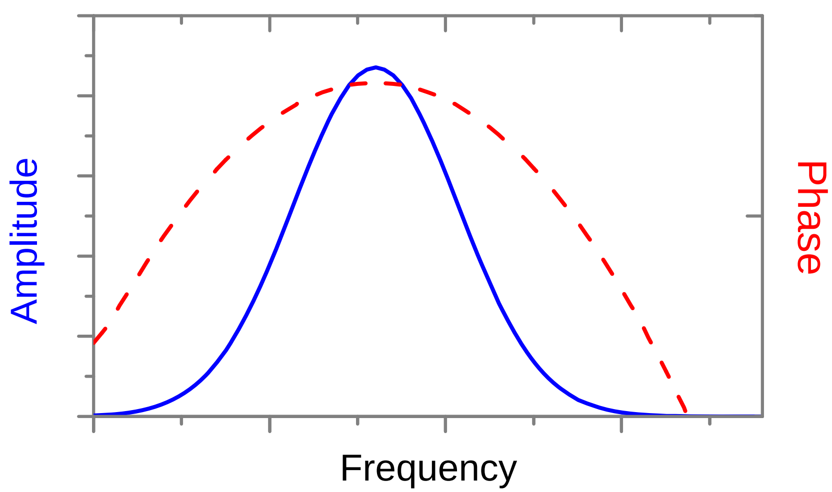
Note: a 1% tail or pedestal in the intensity is 10% in the field strength

Chirp

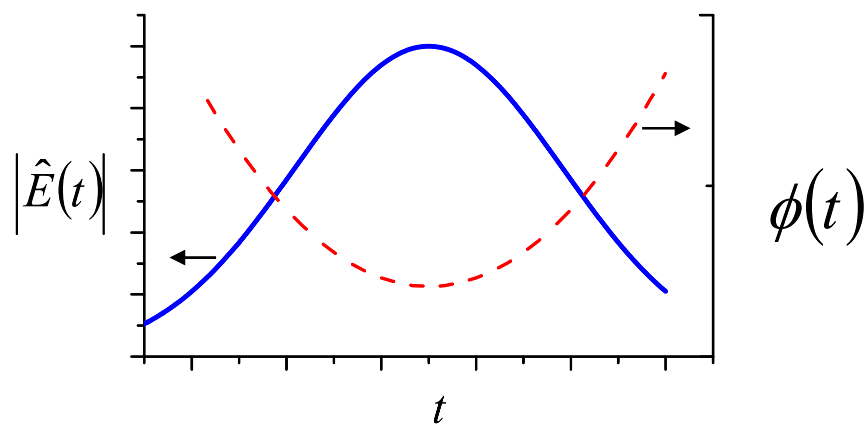


Remove linear phase
(= carrier frequency)





$$E(t)=\overbrace{\left|\hat{E}(t)\right|e^{i\phi(t)}}^{\text{envelope}}\overbrace{e^{i\omega_c t}}^{\text{carrier}}$$



Linear phase removed (pulse center t = 0)

Choice of carrier frequency

Not so good in general (fine for simple pulses):

- Peak of spectrum
- Frequency at peak of temporal profile
- Minimize variation in $\phi(t)$

Most general (and consistent in time and frequency domains)
intensity weighted average frequency:

$$\omega_c = \frac{\int_{-\infty}^{\infty} \left|\hat{E}(t)\right|^2 \omega(t)\,dt}{\int_{-\infty}^{\infty} \left|\hat{E}(t)\right|^2 dt} = \frac{\int_{-\infty}^{\infty} \left|\hat{E}(\Omega)\right|^2 \Omega\,dt}{\int_{-\infty}^{\infty} \left|\hat{E}(\Omega)\right|^2 dt}$$

Random note:

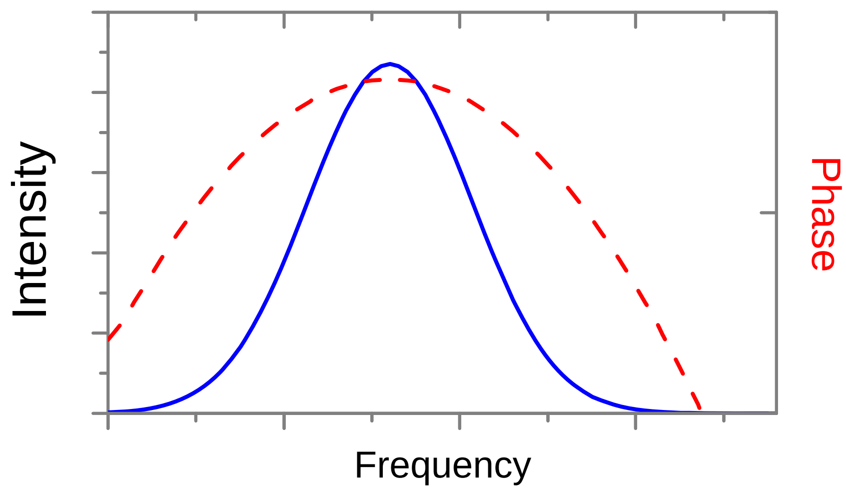
Make sure you write down a physically realistic field

When writing

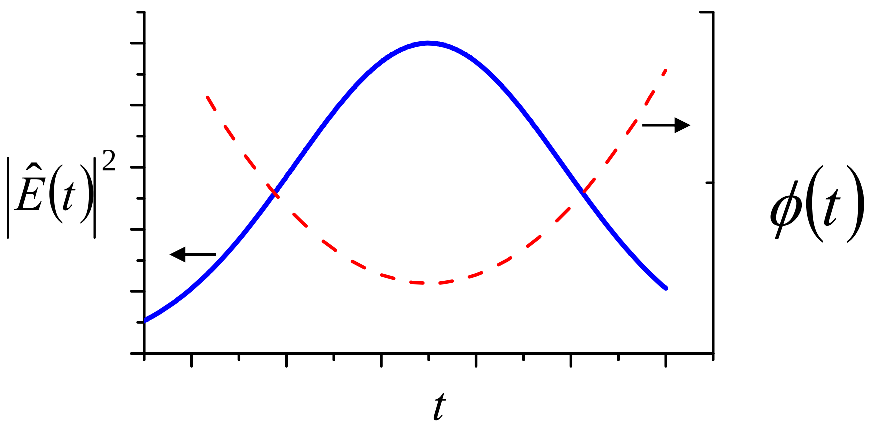
$$E(t) = \hat{E}(t)\cos(\omega t + \phi)$$

The envelope cannot be arbitrarily short, optical elements don't work at DC

Most common spectral and temporal represenations



Linear phase removed (pulse center t = 0)



Linear phase removed (carrier)

Overall (absolute or carrier-envelope) phase set to 0

Chirp: time domain or frequency domain?

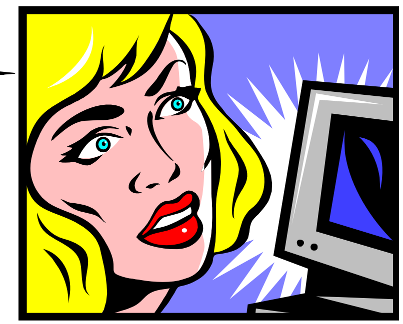
I think my pulses may be chirped, perhaps it should be included in the simulations

No problem:

$$E(t)=\hat{E}(t)\exp\left(i\left[\omega t+\alpha t^{2}\right]\right)$$

No, silly theorist, you have to include it in the frequency domain

Experimentalist



Theorist

Why do you annoying experimentalists have to make everything hard?

Who do you agree with?

Chirp: time domain or frequency domain?

Adding chirp in the time domain changes the spectral width, but not the temporal width of the pulse

$$E(t) = \hat{E}(t)\exp\left(i\left[\omega t + \alpha t^2\right]\right)$$

→ Characteristic of nonlinear propagation

Adding chirp in the frequency domain changes the temporal width, but not the spectral width of the pulse

$$E(\omega) = \hat{E}(\omega)\exp\left(i\beta\omega^2\right)$$

→ Characteristic of linear propagation through dispersive material

Transform limited pulses and time-bandwidth product

For a given bandwidth, the shortest pulse is obtained when the spectral phase is constant → this is known as a "Fourier transform limited pulse"

The product of the bandwidth and the temporal duration is a unitless number known as the "time-bandwidth product", its transform limited value depends on pulse shape

Shape	Intensity profile	τ_p (FWHM)	Spectral profile	$\Delta\omega_p$ (FWHM)	TBP
Gaussian	$e^{-2(t/\tau_G)^2}$	$1.177\,\tau_G$	$e^{-(\Omega\tau_G)^2/2}$	$2.355/\tau_G$	0.441
sech	$\mathrm{sech}^2(t/\tau_s)$	$1.763\ \tau_s$	$\mathrm{sech}^2(\pi\Omega\tau_s/2)$	$1.122/\tau_s$	0.315
Lorentz	$\left[1+(t/\tau_L)^2\right]^{-2}$	$1.287\ \tau_L$	$e^{-2\lvert\Omega\rvert\tau_L}$	$0.693/\tau_L$	0.142
rectangular	1 for $\lvert t/\tau_r\rvert < 1$ 0 elsewhere	τ_r	$\mathrm{sinc}^2(\Omega\tau_R)$	$2.78/\tau_r$	0.443

Gaussian and sech are most important pulse shapes

→ interesting note, they are the only two with the same temporal and spectral profile functional forms

Phase velocity vs. group velocity

The peaks of the e-field for a monochromatic wave move at the phase velocity

$$E(x,t) = |E| e^{i(kx - \omega t)}$$

Peak occurs at $kx - \omega t = n2\pi \rightarrow v_p = x/t = \omega/k$.

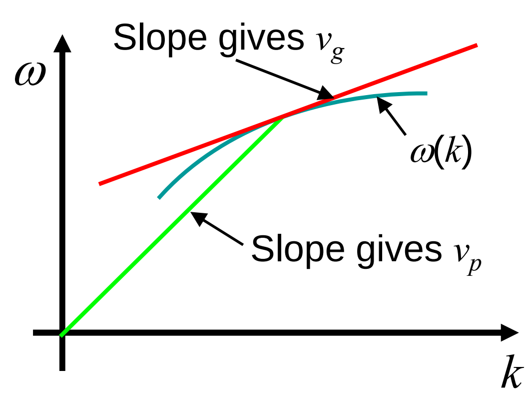
The relationship between ω and k is determined by the index of refraction, n

A pulse occurs due to constructive interference among many "monochromatic" waves

Constructive interference → all waves have same phase, consider two:

$$k_1 x - \omega_1 t = k_2 x - \omega_2 t$$

$$x(k_1 - k_2) = t(\omega_1 - \omega_2) \rightarrow v_g = \frac{x}{t} = \frac{(\omega_1 - \omega_2)}{(k_1 - k_2)} \Rightarrow \frac{d\omega}{dk}$$



Zero group velocity dispersion (GVD) $\frac{\partial v_g}{\partial \omega} = 0$

does not require $\frac{\partial v_p}{\partial \omega} = 0$

and since $v_p = \frac{c}{n}$ thus neither does it require $\frac{\partial n}{\partial \omega} = 0$

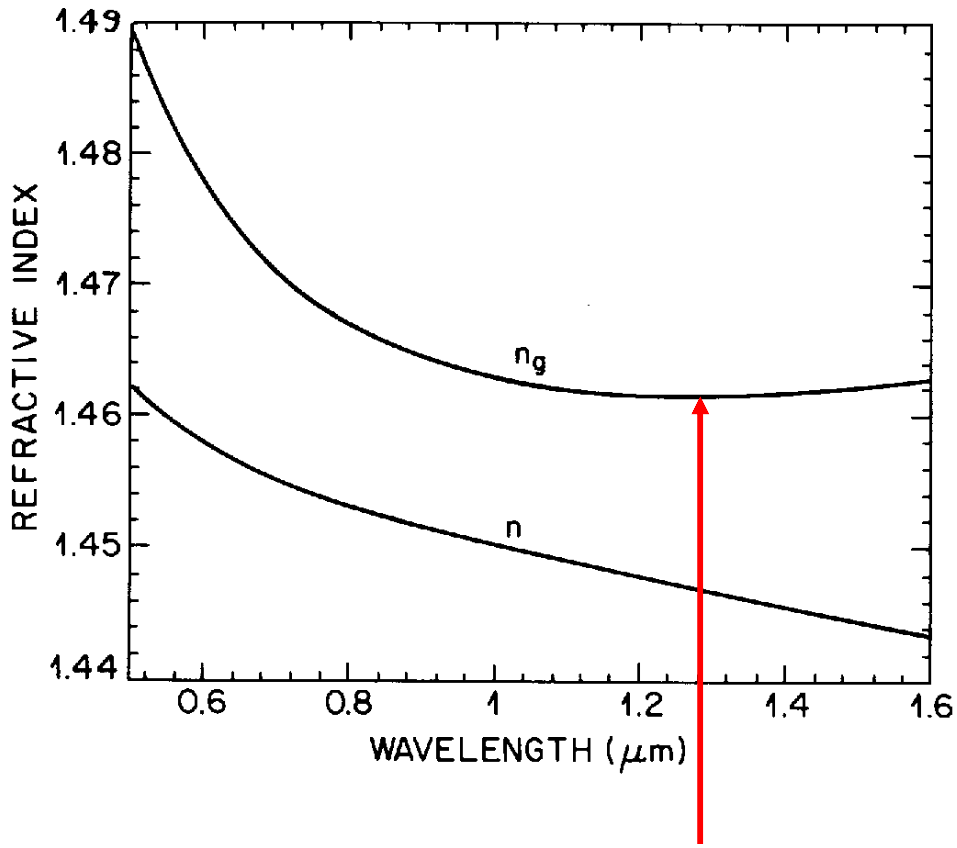
index of refraction

Group and phase velocity (index) – real numbers

$$v_p = \frac{c}{n_p}$$

$$v_g = \frac{c}{n_g}$$

Fused silica



$$\frac{\partial n_g}{\partial \omega} = 0, \quad \frac{\partial n_p}{\partial \omega} \neq 0$$

Propagation – Time Domain & Polarization approach

The propagation of the vector electric field is described by a wave-equation derived from Maxwell's equation:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(x,y,z,t)=\mu_0\frac{\partial^2}{\partial t^2}\mathbf{P}(x,y,z,t)$$

The polarization is induced in the medium by the electric field. Commonly:

- move non resonant part into an index of refraction
- separate into linear and nonlinear parts

To stay completely general, for now we just assume we know how to calculate P.

Simplify life by only considering one spatial dimension

$$\frac{\partial^2}{\partial z^2}E(z,t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}E(z,t)=\mu_0\frac{\partial^2}{\partial t^2}P(z,t)$$

Propagation – Time Domain & Polarization approach II

Decompose E and P into envelope and carrier:

$$E(z,t) = \hat{E}(z,t) e^{-i(kz-\omega t)} \qquad P(z,t) = \hat{P}(z,t) e^{-i(kz-\omega t)}$$

Calculate derivatives:

$$\frac{\partial^2 E(z,t)}{\partial z^2} = \frac{\partial^2 \hat{E}(z,t)}{\partial z^2} e^{-i(kz-\omega t)} - 2ik \frac{\partial \hat{E}(z,t)}{\partial z} e^{-i(kz-\omega t)} - k^2 \hat{E}(z,t) e^{-i(kz-\omega t)}$$

$$\frac{\partial^2 E(z,t)}{\partial t^2} = \frac{\partial^2 \hat{E}(z,t)}{\partial t^2} e^{-i(kz-\omega t)} - 2i\omega \frac{\partial \hat{E}(z,t)}{\partial t} e^{-i(kz-\omega t)} - \omega^2 \hat{E}(z,t) e^{-i(kz-\omega t)}$$

$$\frac{\partial^2 P(z,t)}{\partial t^2} = \frac{\partial^2 \hat{P}(z,t)}{\partial t^2} e^{-i(kz-\omega t)} - 2i\omega \frac{\partial \hat{P}(z,t)}{\partial t} e^{-i(kz-\omega t)} - \omega^2 \hat{P}(z,t) e^{-i(kz-\omega t)}$$

Transform into frame of reference (z',t') moving at c, from chain rule:

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t'}$$

$$\frac{\partial^2}{\partial z^2} E(z,t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E(z,t) = \mu_0 \frac{\partial^2}{\partial t^2} P(z,t)$$

$$\frac{\partial}{\partial z} \rightarrow \frac{\partial}{\partial z'} - \frac{1}{c} \frac{\partial}{\partial t'}$$

Propagation – Time Domain & Polarization approach III

Assume $\frac{\partial \hat{E}}{\partial z} << 2k$

Cancel carrier $e^{-i(kx-\omega t)}$ in all terms

Yielding:

$$-2ik\frac{\partial \hat{E}(z',t')}{\partial z'} - \frac{1}{c^2}\frac{\partial^2 \hat{E}(z',t')}{\partial t'^2} = \mu_0\left[\frac{\partial^2 \hat{P}(z',t')}{\partial t'^2} + 2i\omega\frac{\partial \hat{P}(z',t')}{\partial t'} - \omega^2 \hat{P}(z',t')\right]$$

Assume that the envelope varies slowly in time compared to the carrier,

i.e., $\frac{\partial(\hat{E}\ or\ \hat{P})}{\partial t} < \omega$ thus we can drop the time derivatives

SVEA – slowly varying envelope approximation

$$2ik\frac{\partial \hat{E}(z',t')}{\partial z'} = \mu_0\omega^2\hat{P}(z',t')$$

Discretize derivative:

$$\hat{E}(z'+\Delta z',t') = \hat{E}(z',t') - \frac{i}{2}\frac{1}{c\varepsilon_0}\omega\hat{P}(z',t')\Delta z$$

Conceptually:

Propagation consists of taking the pulse, calculating the polarization in induces, the polarization radiates, which is then added to the original field

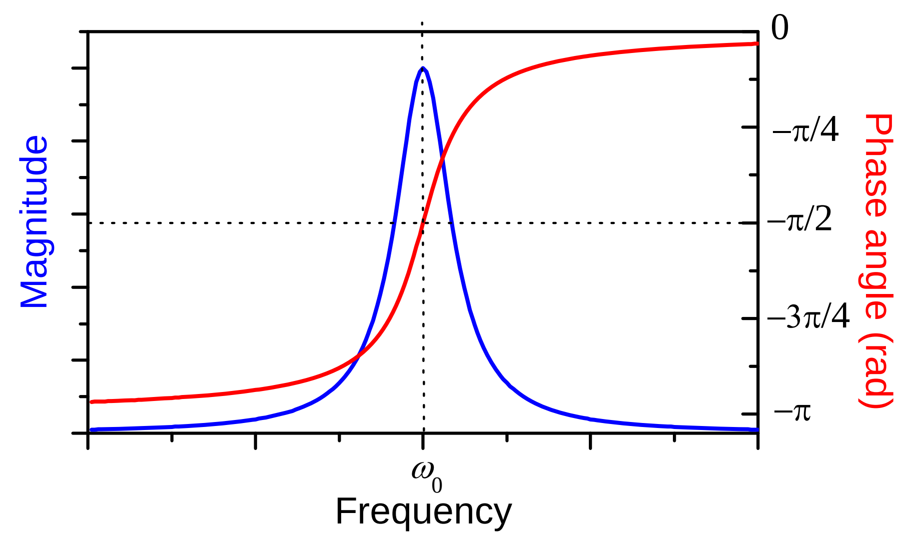
Phase of polarization: absorption vs. refraction

Based on this simple equation we can immediately understand absorption versus refraction.

Consider a simple Lorentz model (electron on a spring) for the polarization, gives amplitude and phase relative to driving field

On resonance: $P \sim -i$ → destructively interferes with E → absorption (less light) due to $-i$ in equation

Off resonance: P 0 or 180 degrees, adds in quadrature to E → index of refraction (phase shift of light) due to $-i$ in equation



$$\hat{E}(z'+\Delta z',t') = \hat{E}(z',t') - \frac{i}{2}\frac{1}{c\varepsilon_0}\omega\hat{P}(z',t')\Delta z$$

Nonlinearity: Kerr effect and Self phase modulation I

Index of refraction depends on intensity:

$$n = n_0 + n_2 I$$

$n_2 \sim 10^{-16}$ to 10^{-15} cm^2/W

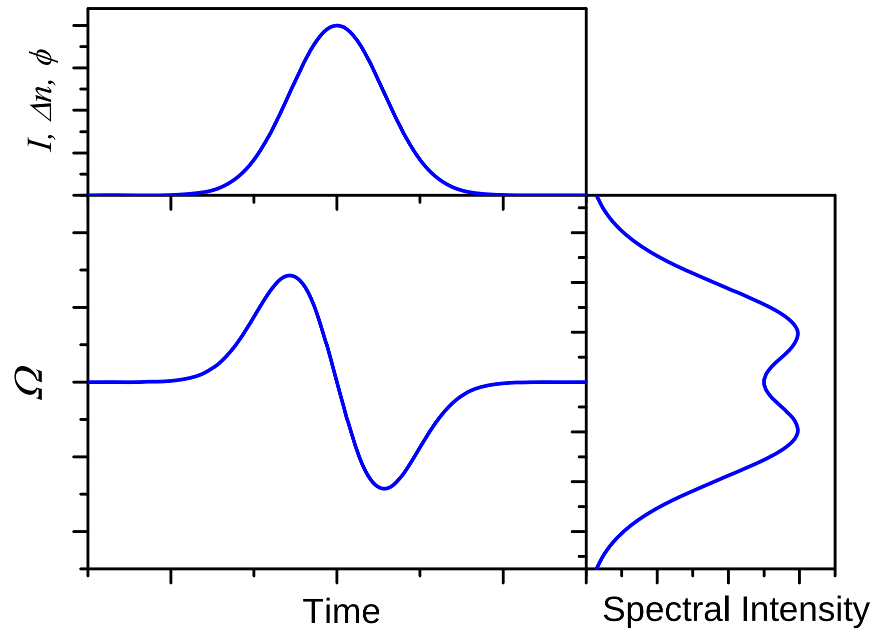
To get $n_2 I = 0.001$ in 10 μm spot need 100 W to 1 kW

If $I(t)$ then $n(t)$ which means $\phi(t)$

Interference between temporal center and wings of pulse result in spectral dip(s).

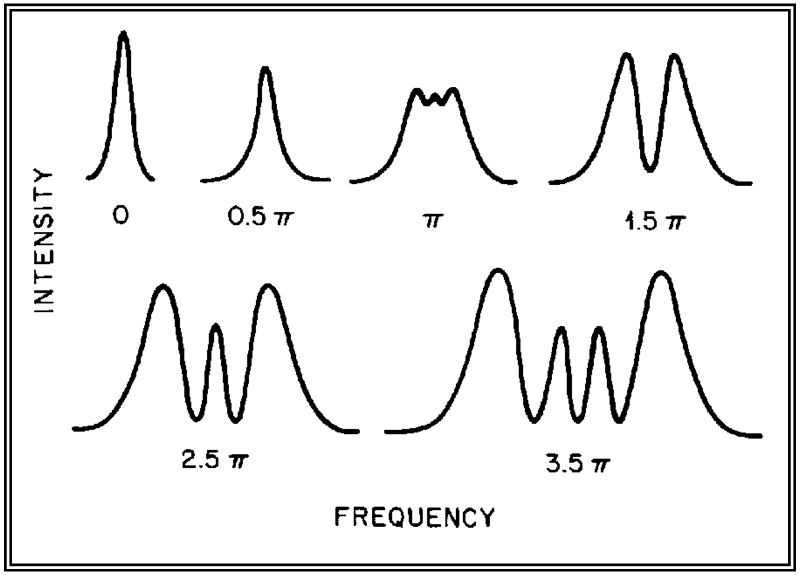
Depth & number of dips give nonlinear phase

Note that spectral width increases

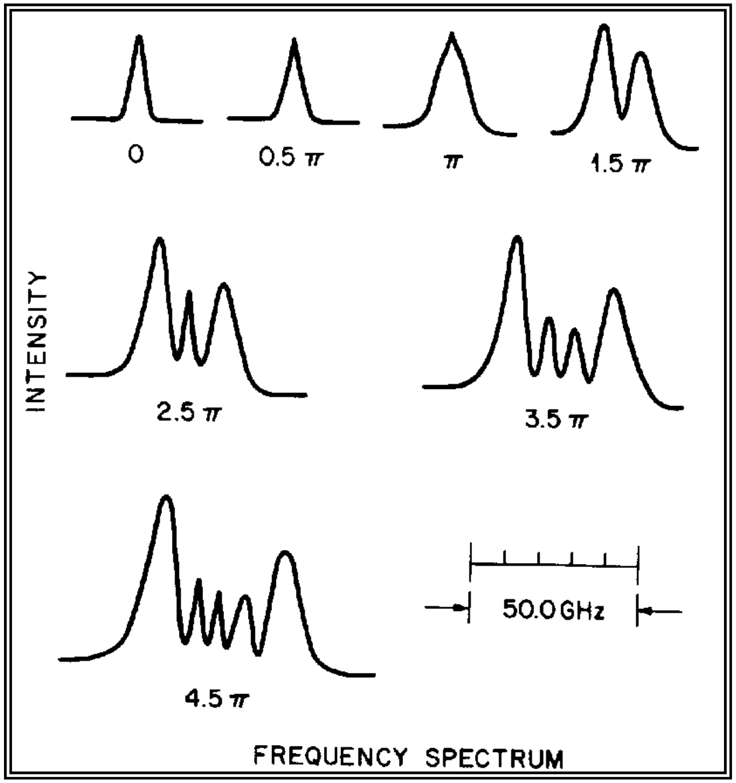


Nonlinearity: Kerr effect and Self phase modulation II

Calculation



Experiment



Labeled by maximum phase shift

Stolen and Lin, Phys. Rev. A **17**, 1448 (1978)

Linear propagation

In free space (and one spatial dimension) the E-field propagates according to

$$\frac{\partial^2}{\partial z^2}E(z,t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}E(z,t)=0$$

From our PDE's course we know a solution, if we have an initial condition (t=0)

$$f(z)$$

Then at later time the solution is just

$$f(z-ct)$$

I.e., the initial "pulse" simply propagates at the velocity c.

What happens when a medium is present?
→ A polarization is induced, through which the medium reacts back on the pulse

The real question is, "How does propagation in a medium differ from free space?"
→ "subtract out" trivial parts to find out how initial pulse is changes it shape

this actually takes some work

Linear propagation in dispersive media I

Start from wave-equation in one spatial dimension

$$\frac{\partial^2}{\partial z^2}E(z,t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}E(z,t)=\mu_0\frac{\partial^2}{\partial t^2}P(z,t)$$

We will assume the P is linearly dependent on E, write it

$$P(t,z)=\varepsilon_0\int_{-\infty}^{t}\chi(t,t')E(t',z)dt'$$

χ is the linear susceptibility of the system, it embodies the fact that the system has memory

We will assume that it only depends on the time difference $t-t'$ – a safe assumption for linear optics (no absolute origin of time), but needs to be remembered in nonlinear optics when an effective linear χ is used.

$$P(t,z)=\varepsilon_0\int_{-\infty}^{t}\chi(t-t')E(t',z)dt'$$

This is a convolution, in the frequency domain: $\tilde{P}(z,\Omega)=\varepsilon_0\chi(\Omega)\tilde{E}(z,\Omega)$

Linear propagation in dispersive media II

Define

$$\varepsilon(\Omega) = 1 + \chi(\Omega)$$

Yielding a frequency-domain wave equation

$$\left[\frac{\partial^2}{\partial z^2} - \frac{\Omega^2}{c^2}\varepsilon(\Omega)\right]\tilde{E}(z,\Omega) = 0$$

The general solution is

$$\tilde{E}(z,\Omega) = \tilde{E}(0,\Omega)e^{-ik(\Omega)z}$$

where

$$k^2(\Omega) = \frac{\Omega^2}{c^2}\varepsilon(\Omega) = \frac{\Omega^2}{c^2}n^2(\Omega)$$

Note: in fiber optics, β is used as the longitudinal propagation constant.

Expand k around carrier frequency

$$k(\Omega) = k_0 + \left.\frac{dk}{d\Omega}\right|_{\omega_0}(\Omega - \omega_0) + \frac{1}{2}\underbrace{\left.\frac{d^2k}{d\Omega^2}\right|_{\omega_0}}_{\beta_2}(\Omega - \omega_0)^2 + \ldots = k_0 + \delta k$$

Linear propagation in dispersive media III

write

$$\tilde{E}(z,\Omega) = \tilde{E}(0,\Omega)e^{-i\delta kz}e^{-ik_0 z}$$

Slowly varying part (in time and space)

Back into the time domain we go….

$$E(z,t) = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \tilde{E}(0,\Omega)e^{-i\delta kz}e^{i(\Omega-\omega_0)t}d\Omega\right\}e^{i(\omega_0 t - k_0 z)}$$

$\hat{E}(z,t)$

$$E(z,t) = \frac{1}{2}\hat{E}(z,t)e^{i(\omega_0 t - k_0 z)}$$

Seek a wave equation for envelope, need to define $\hat{P}$

$$\tilde{P}(z,\Omega) = \varepsilon_0\left(\varepsilon(\omega_0) - 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\frac{d^n\varepsilon}{d\Omega^n}\bigg|_{\omega_0}(\Omega-\omega_0)^n\right)\tilde{E}(z,\Omega) \qquad \varepsilon^{(n)}(\omega_0) = \frac{\partial^n\varepsilon}{\partial\Omega^n}\bigg|_{\omega_0}$$

$$P(z,t) = \frac{1}{2}\left\{\varepsilon_0[\varepsilon(\omega_0 - 1)]\hat{E}(z,t) + \varepsilon_0\sum_{n=1}^{\infty}(-i)^n\frac{\varepsilon^{(n)}(\omega_0)}{n!}\frac{\partial^n}{\partial t^n}\hat{E}(z,t)\right\}e^{i(\omega_0 t - k_0 z)}$$

Linear propagation in dispersive media IV

Go into moving frame at $v_g = \left(\left. \frac{dk}{d\Omega} \right|_{\omega_0} \right)^{-1}$

$$z' = z, \quad t' = t - \frac{z}{v_g}$$

Yielding:

$$\frac{\partial}{\partial z'}\hat{E} - \frac{i}{2}k_0'' \frac{\partial^2}{\partial t'^2}\hat{E} + D = -\frac{i}{2k_0}\frac{\partial}{\partial z'}\left(\frac{\partial}{\partial z'} - \frac{2}{v_g}\frac{\partial}{\partial t'} \right)\hat{E}$$

Where higher order dispersion has been lumped into

$$D = \frac{i}{3k_0c^2}\sum_{n=3}^{\infty}\frac{(-i)^n}{n!}\left[\omega_0^2\varepsilon^{(n)}(\omega_0) + 2n\omega_0\varepsilon^{(n-1)}(\omega_0) + n(n-1)\varepsilon^{(n-2)}(\omega_0)\right]\frac{\partial^n}{\partial \eta^n}\hat{E}$$

and

$$k_0'' = \left. \frac{\partial^2 k}{\partial \Omega^2} \right|_{\omega_0} = -\frac{1}{v_g^2}\left. \frac{dv_g}{d\Omega} \right|_{\omega_0}$$

$$= \frac{1}{2k_0}\left[\frac{2}{v_g^2} - \frac{2}{c^2}\varepsilon(\omega_0) - \frac{4\omega_0}{c^2}\varepsilon^{(1)}(\omega_0) - \frac{\omega_0^2}{c^2}\varepsilon^{(2)}(\omega_0) \right]$$

Group Velocity Dispersion (GVD)

Linear propagation in dispersive media V

Again, further assumption on slowness of envelope:

$$\left|\frac{1}{k_0}\left(\frac{\partial}{\partial z'} - \frac{2}{v_g}\frac{\partial}{\partial t'}\right)\hat{E}\right| = \left|\frac{1}{k_0}\left(\frac{\partial}{\partial z} - \frac{2}{v_g}\frac{\partial}{\partial t}\right)\hat{E}\right| << \left|\hat{E}\right|$$

Allows the right side to be set to zero.

Also, ignore high order dispersion, yielding

$$\frac{\partial}{\partial z'}\hat{E} - \frac{i}{2}k_0''\frac{\partial^2}{\partial t'^2}\hat{E} = 0$$

This is a wave equation for the envelope in the frame moving with it

→ For no GVD, the envelope does not change

With GVD this can be directly solved:

In time: $$\hat{E}(t,z) = \frac{1}{\sqrt{2\pi i k'' z}}\int_{-\infty}^{t}\hat{E}(t', z=0)\exp\left(i\frac{(t-t')^2}{2k_0'' z}\right)dt'$$

In frequency: $$\hat{E}(\Omega, z) = \hat{E}(\Omega, 0)e^{-\frac{i}{2}k''\Omega^2 z}$$

Linear propagation in dispersive media VI

$$\hat{E}(\Omega, z) = \hat{E}(\Omega, 0)e^{-\frac{i}{2}k''\Omega^2 z}$$

We see that GVD imposes a quadratic phase (linear frequency chirp) that increases with propagation distance

GVD I

Dispersion terms in frequency and wavelength

$$\frac{dk}{d\Omega}=\frac{n}{c}+\frac{\Omega}{c}\frac{dn}{d\Omega}=\frac{1}{c}\left(n-\lambda\frac{dn}{d\lambda}\right)=\frac{1}{v_g}$$

$$\frac{d^2k}{d\Omega^2}=\frac{2}{c}\frac{dn}{d\Omega}+\frac{\Omega}{c}\frac{d^2n}{d\Omega^2}=\left(\frac{\lambda}{2\pi c}\right)\frac{1}{c}\left(\lambda^2\frac{d^2n}{d\lambda^2}\right)=\frac{2\pi c}{\Omega^2 v_g^2}\frac{dv_g}{d\lambda}$$

GVD

$$\frac{d^3k}{d\Omega^3}=\frac{3}{c}\frac{d^2n}{d\Omega^2}+\frac{\Omega}{c}\frac{d^3n}{d\Omega^3}=-\left(\frac{\lambda}{2\pi c}\right)^2\frac{1}{c}\left(3\lambda^2\frac{d^2n}{d\lambda^2}+\lambda^3\frac{d^3n}{d\lambda^3}\right)$$

3rd order dispersion
or
GVD slope

GVD is usually characterized by either

$$\frac{d^2k}{d\Omega^2}\equiv k_0''\equiv\beta_2\qquad\left[\frac{t^2}{l}\right]$$

"physicist's parameter"

or

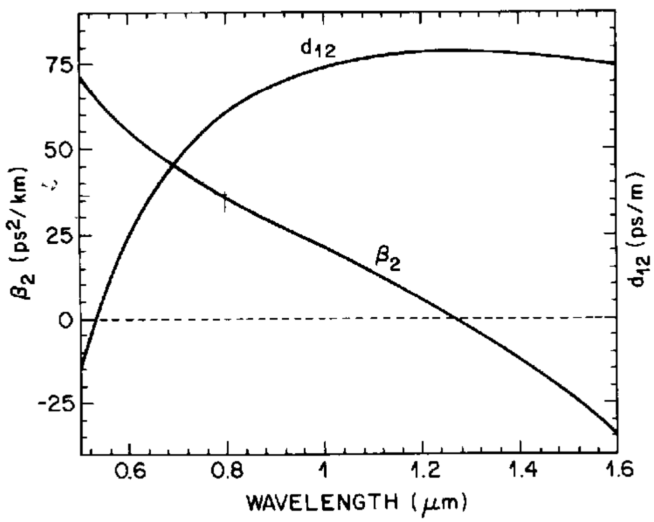
$$D=\frac{d^2k}{d\lambda^2}=-\frac{2\pi c}{\lambda^2}\frac{d^2k}{d\Omega^2}=-\frac{\lambda}{c}\frac{d^2n}{d\lambda^2}\qquad\left[\frac{t}{\lambda\cdot l}\right]$$

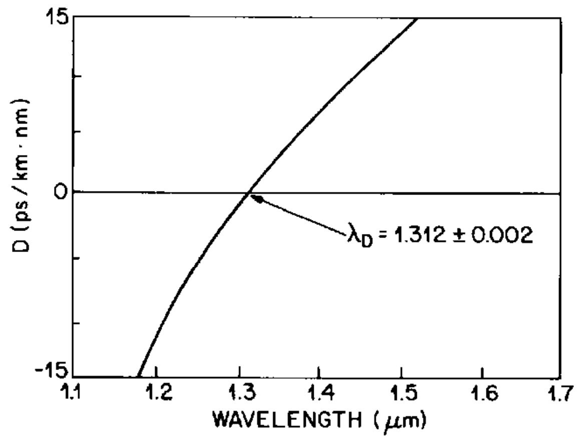
"engineer's parameter"

GVD II

Normal dispersion: $k_0'' > 0, \quad D < 0$

Anomalous dispersion: $k_0'' < 0, \quad D > 0$





Define dispersive lengths

$$L_D = \frac{\tau^2}{|k_0''|}$$

An initially unchirped Gaussian doubles length in $0.6\,L_D$

$$0.6\frac{(10\text{ fs})^2}{35\,\frac{\text{fs}^2}{\text{mm}}} = 1.8\,\text{mm}$$

$$L_D' = \frac{1}{|k_0''|\Delta\omega_p^2}$$

An initially unchirped pulse broadens in L_D'

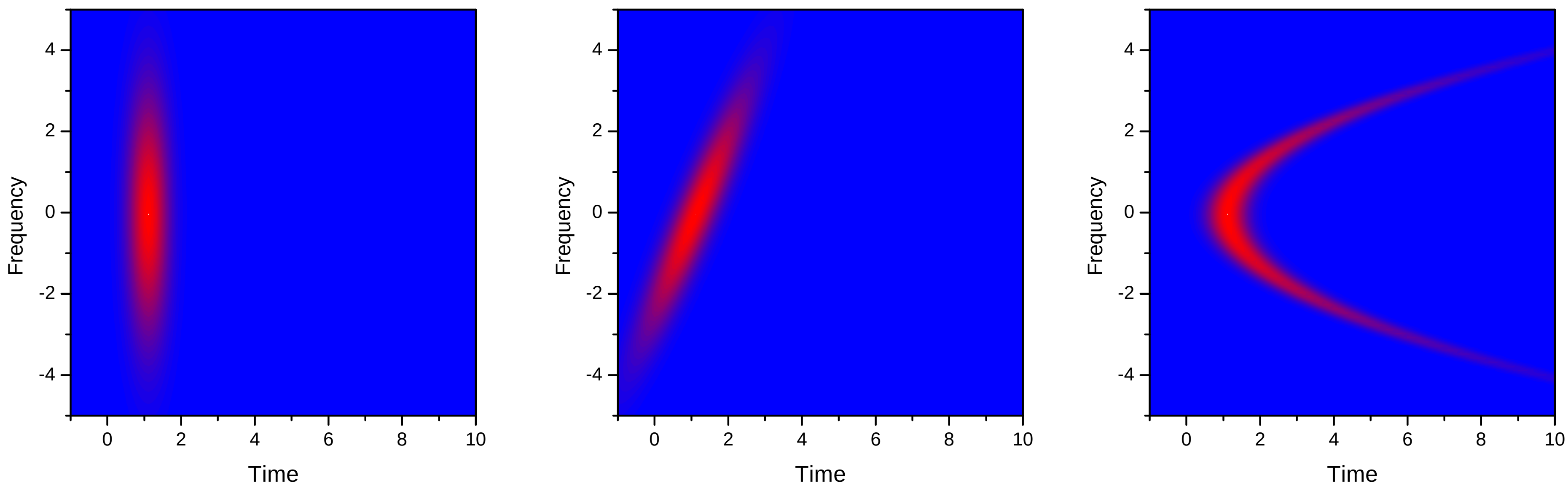
Third order dispersion

Change in GVD as a function of frequency/wavelength (thus “GVD slope”)

Most important when GVD itself is small

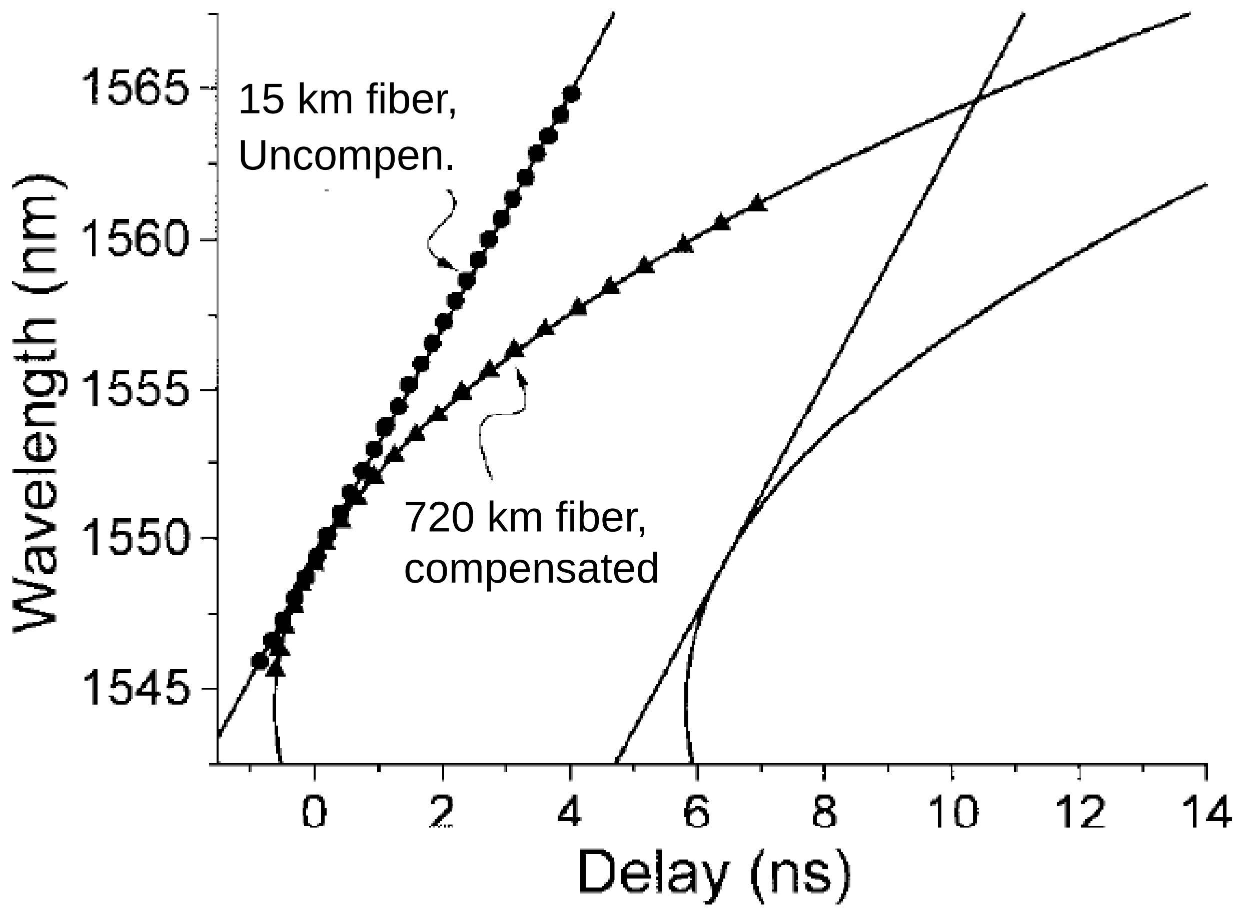
Easiest to understand by plotting intensity(t,freq or wavelength)

Transform limited | GVD – linear chirp | Third order



Third order dispersion II

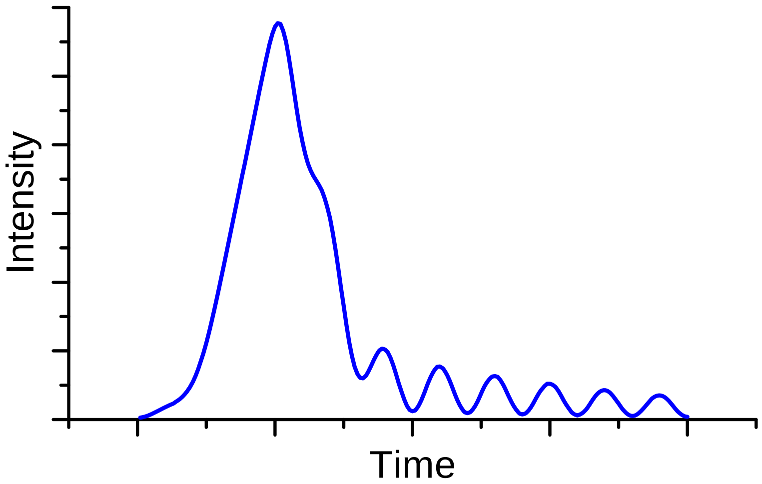
Accumulates to measurable time delays in optical fiber.

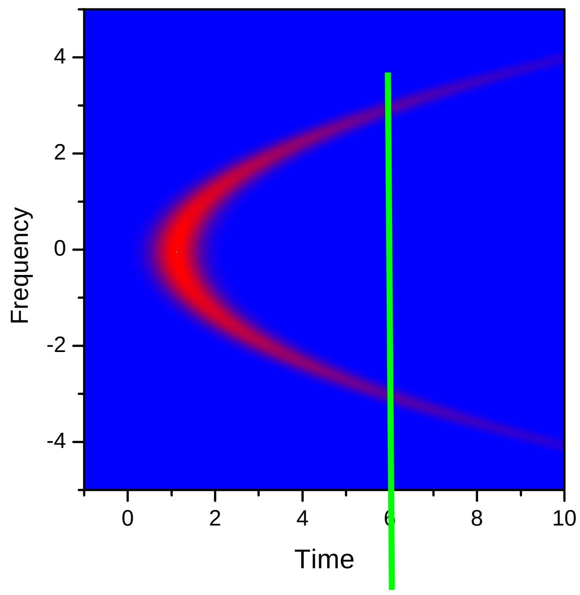


Cundiff, Collings, Boivin, Nuss, Bergman, Knox, and Evangelides, J. Lightwave Technol. 17, 811 (1999)

Third order dispersion III

Effect on pulse:





Tail with oscillations

Oscillations occur due interference
between overlapping wings of pulse

Modeling

1) Directly solve Maxwell's equations
   Material included as polarization from resonance
   Uggh…

2) Solve envelope equations
   include term for nonlinearity → nonlinear Schroedingers equations
   solitons

3) Split-step Fourier transform
   Dispersion is most easily included in frequency domain
   Nonlinearity is most easily included in time domain
   Switch back and forth between them using Fourier Transforms