Optical Frequency Spectrum of a Mode-locked Laser

What is the frequency spectrum of a mode-locked laser?
Two approaches:

1) Modes of cavity
   Mode-locking merely imposes fixed phase relationship → emission spectrum should match cavity spectrum

   Care needed in including effects of dispersion

2) Calculate spectrum of output pulse train
   Simply Fourier transform

   Need to be careful to consider electric field
   Derived from considering group and phase velocities inside laser

Cavity mode approach

Cavity modes: integer number of wavelengths in one roundtrip

$$\omega_m = \frac{c}{nL} m$$

Where n is the “effective” index of refraction and L is the cavity length

The mode spacing is

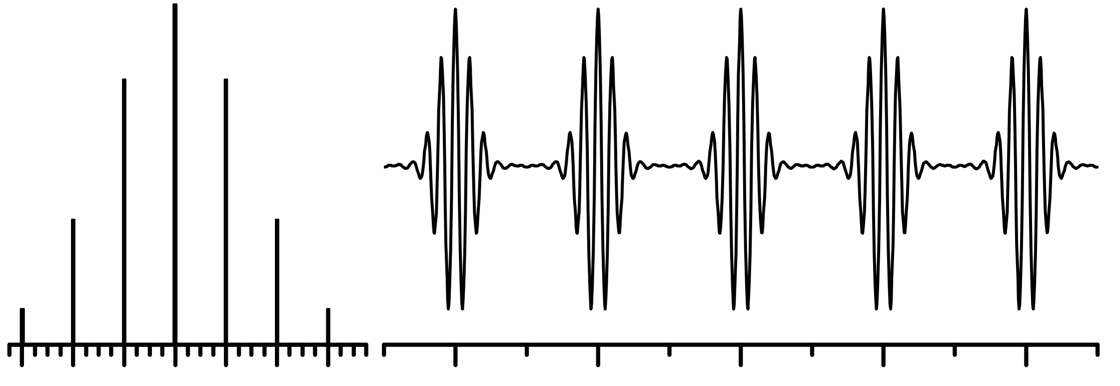
$$\Delta\omega = \frac{c}{nL}$$

This means we describe the pulse train by simply summing up the modes with frequency ω_m.

→ Integer multiples of $\Delta\omega$

→simple Fourier series

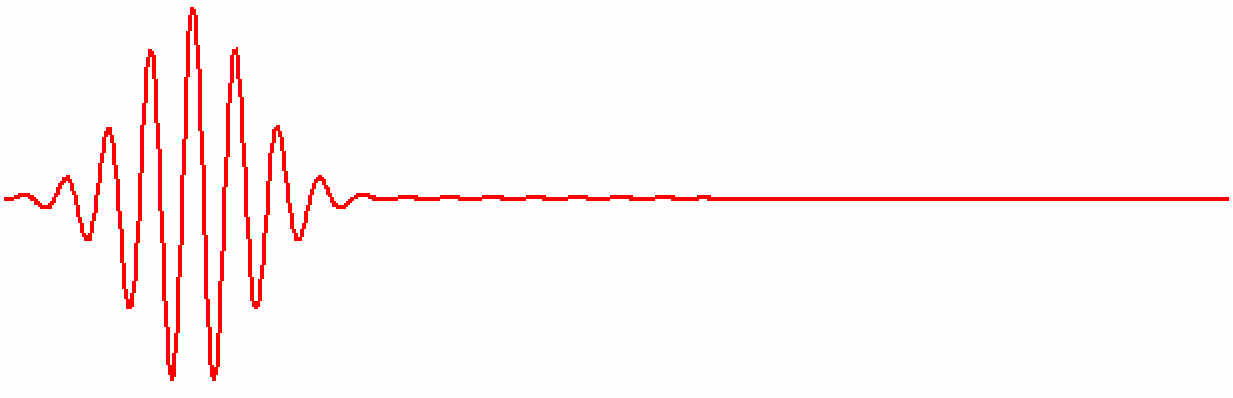
→ time domain is periodic at $1/\Delta\omega$



Time domain approach

Consider evolution of pulse in cavity in terms of envelope and carrier

They will propagate at different rates due to difference between phase and group velocities

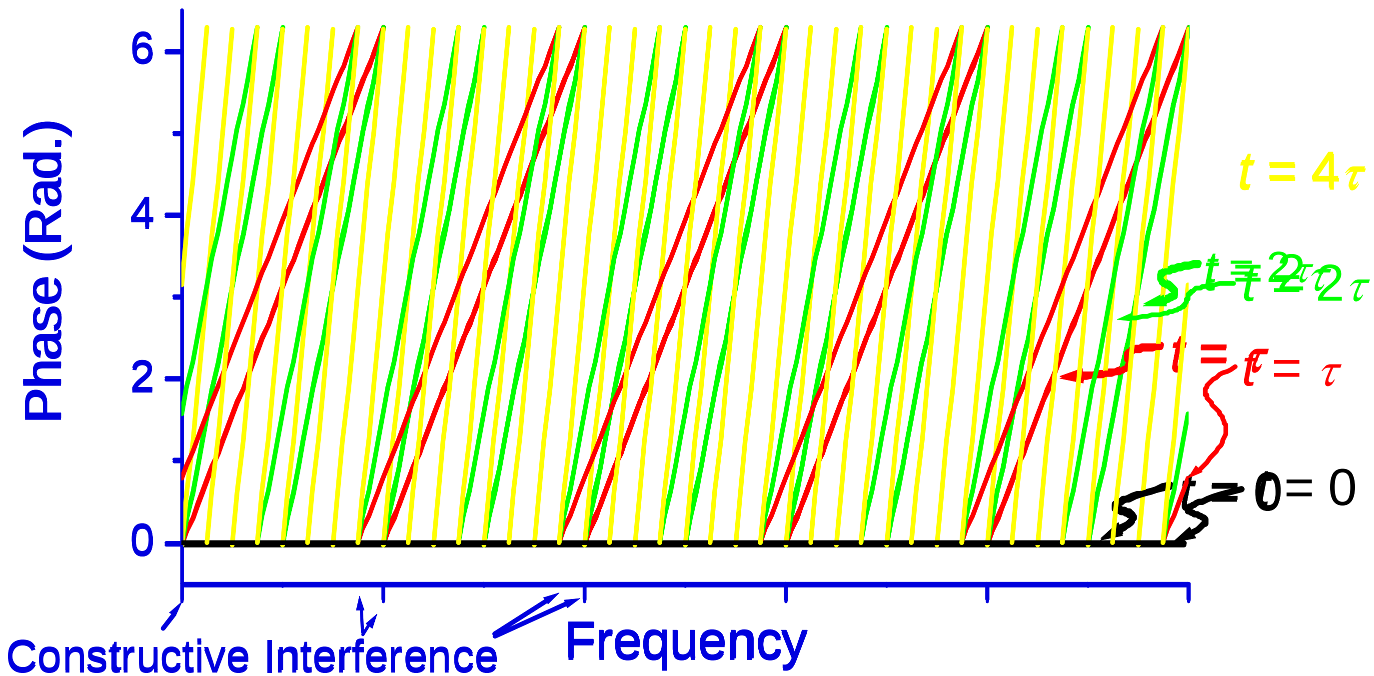


As this happens in the cavity, but not outside, it means each pulse in the train has a unique phase:

Effect of pulse phase on spectrum I

First, pictorially

(recall, a shift in time = linear phase with frequency)



With pulse-to-pulse phase shift

Effect of pulse phase on spectrum II

The electric field of one pulse is:

$$E_{pulse}(t) = \hat{E}(t)\, e^{i(\omega_c t + \phi_0)}$$

Mathematically, write pulse train with pulse period τ ($t \rightarrow t - n\tau$, and sum):

$$E(t) = \sum_n \hat{E}(t - n\tau) e^{i(\omega_c t - n\omega_c \tau + n\Delta\phi + \phi_0)} = \sum_n \hat{E}(t - n\tau) e^{i(\omega_c t + n(\Delta\phi - \omega_c \tau) + \phi_0)}$$

Take Fourier transform

$$E(\omega) = \int \sum_n \hat{E}(t - n\tau) e^{i(\omega_c t + n(\Delta\phi - \omega_c \tau) + \phi_0)} e^{-i\omega t} dt = \sum_n e^{i(n(\Delta\phi - \omega_c \tau) + \phi_0)} \int \hat{E}(t - n\tau) e^{-i[(\omega - \omega_c)t]} dt$$

Let

$$\tilde{E}(\omega) = \int \hat{E}(t) e^{-i\omega t} dt$$

Recall identity

$$\int f(x - a) e^{-i\alpha x} dx = e^{-i\alpha a} \int f(x) e^{-i\alpha x} dx$$

Yielding

$$E(\omega) = \sum_n e^{i(n(\Delta\phi - \omega_c \tau) + \phi_0)} e^{-in(\omega - \omega_c)\tau} \tilde{E}(\omega - \omega_c) = e^{i\phi_0} \tilde{E}(\omega - \omega_c) \sum_n e^{i(n\Delta\phi - n\omega\tau)}$$

Effect of pulse phase on spectrum III

Use the Poisson formula

$$\sum_{m=-\infty}^{\infty} f\left(x-mp\right)=\sum_{k=-\infty}^{\infty}\frac{1}{p}F\left(\frac{k}{p}\right)e^{2\pi ikx/p}$$

(in reverse) to obtain

$$E\left(\omega\right)=e^{i\phi_0}\tilde{E}\left(\omega-\omega_c\right)\sum_n e^{i\left(n\Delta\phi-n\omega\tau\right)}$$

$$=e^{i\phi_0}\tilde{E}\left(\omega-\omega_c\right)\sum_m \delta\left(\Delta\phi_{ce}-\omega\tau-2\pi m\right)$$

This is a series of delta functions with frequencies

$$\omega_m=\frac{2m\pi}{\tau}-\frac{\Delta\phi_{ce}}{\tau}$$

Or in non-angular frequencies ($f_r = 1/\tau$):

$$\nu_m = m\, f_r + f_0$$

where $f_0 = -\Delta\phi \cdot f_r/2\pi$

This confirms our simple pictorial analysis that the pulse-to-pulse phase shift results in an offset of the comb from integer multiples of f_r

Choice of carrier frequency

The choice of carrier frequency is slightly more constrained than for a single pulse, require that

$$\nu_c t = 2\pi m$$

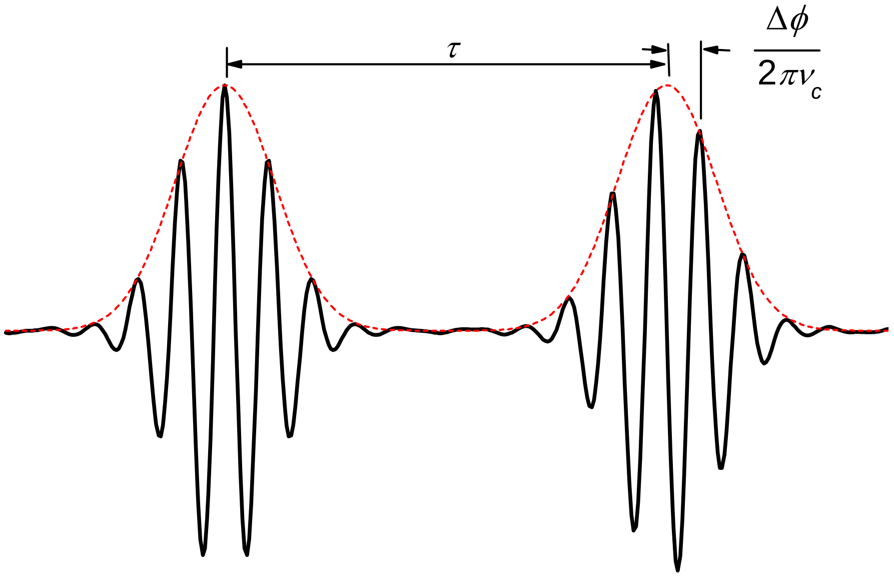
Where t is the time between peaks of the carrier wave

$$t = \tau + \frac{\Delta\phi}{2\pi\nu_c}$$

Which gives

$$_m\nu_c = \frac{m}{\tau} - \frac{\Delta\phi}{2\pi\tau} = mf_{rep} + f_0$$

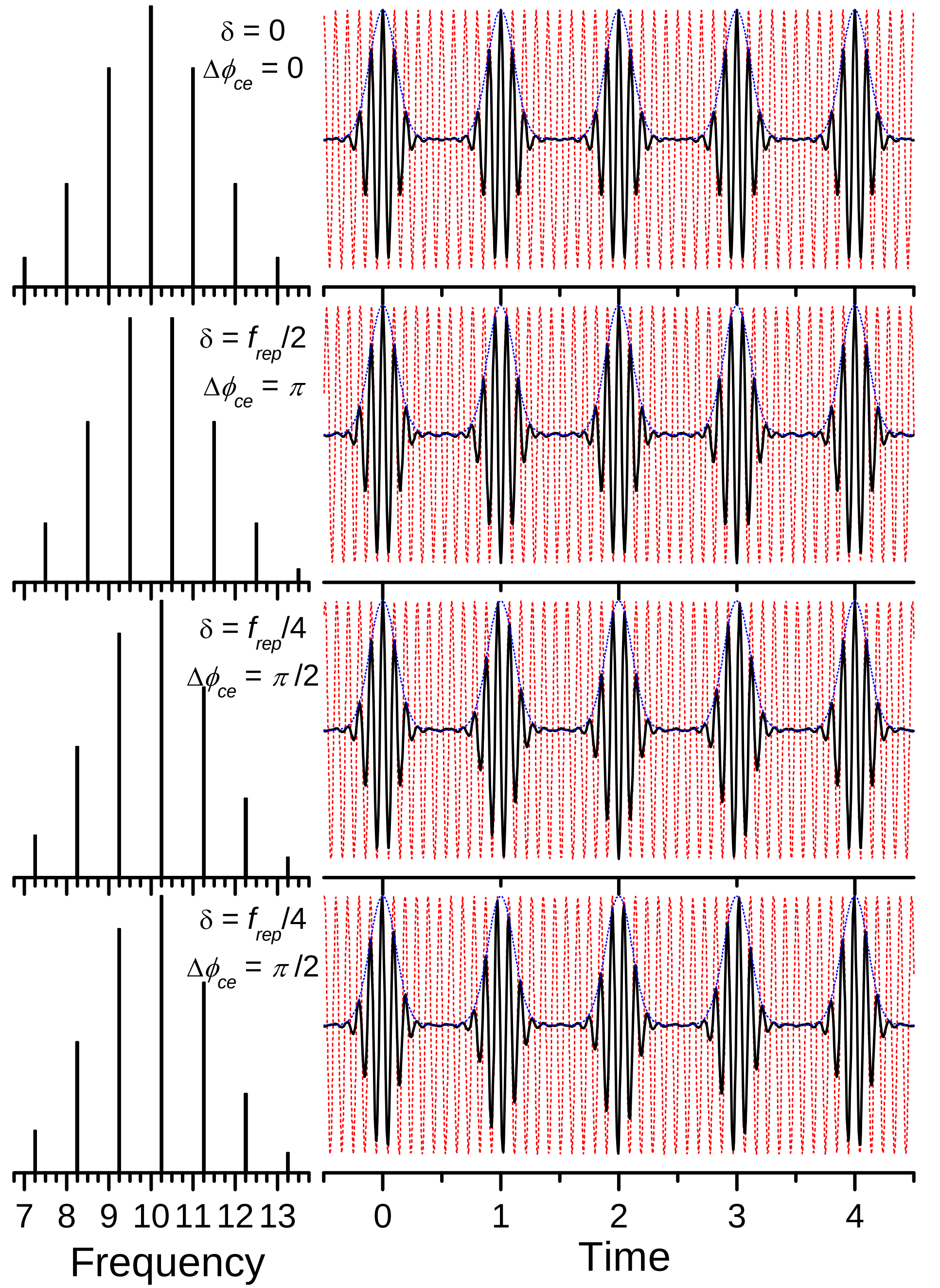
➔ The carrier must be a comb line



Effect of pulse phase on spectrum IV

Simple verification:

Add up sine-waves



Overall phase shift of $\pi/8$

no change to spectrum

does change time domain

Reconciling modes with time-domain picture I

Hey, wait a minute, what about the result from just considering the cavity modes?

We didn't consider dispersion – and there must be dispersion if the group and phase velocities differ from one another.

Since the pulse hangs together, there must be no group velocity dispersion, but there can be phase velocity dispersion

No GVD requires

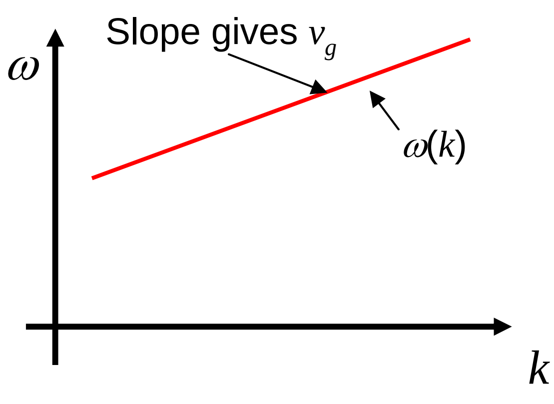
$$\frac{\partial k}{\partial \omega} = \frac{1}{v_g} = const.; \quad \frac{\partial^m k}{\partial \omega^m} = 0 \quad \text{for } m > 1$$

This is satisfied by

$$k = k_0 + \frac{1}{v_g}\omega$$

Which corresponds to

$$n = c\left(\frac{1}{v_g} + \frac{k_0}{\omega}\right)$$



Reconciling modes with time-domain picture II

If we plug

$$n = c\left(\frac{1}{v_g} + \frac{k_0}{\omega}\right)$$

Into the frequency of the Fabry-Perot modes

$$\omega_m = \frac{2\pi c}{nL} m$$

And solve for the frequencies we get

$$\omega_m = \frac{2\pi v_g}{L} m - k_0 v_g$$

Which shows that there is a frequency offset when dispersion is included, i.e., both pictures are consistent

Origin of offset frequency

Using this result together with

$$k_0 = \omega_c v_g \left(\frac{1}{v_p} - \frac{1}{v_g} \right)$$

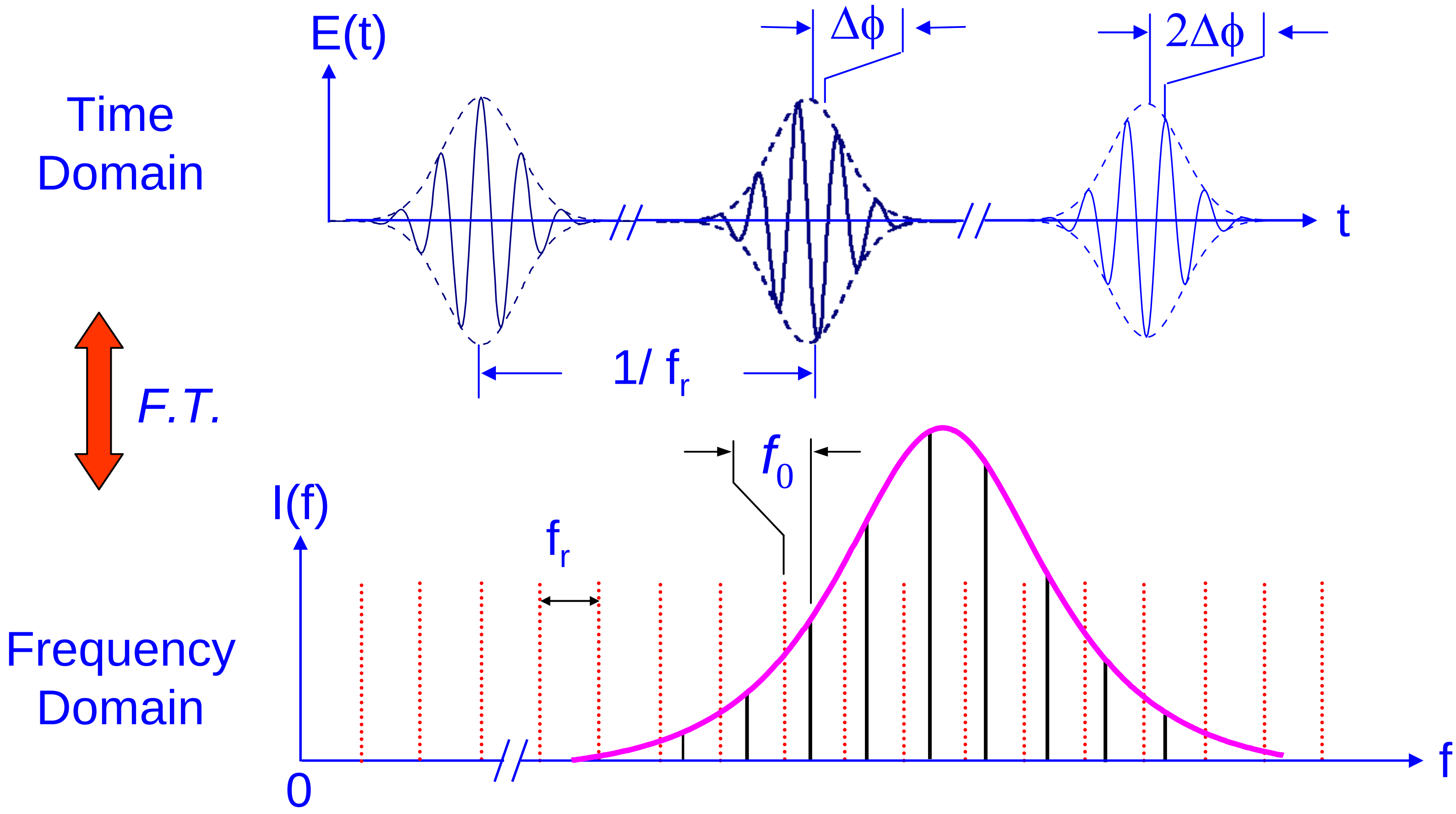
We get

$$f_0 = \frac{\omega_c v_g}{2\pi} \left(\frac{1}{v_g} - \frac{1}{v_p} \right)$$

This explicitly shows how the offset depends on the difference between phase and group velocities

This is a bit troubling – it seems that f_0 depends on choice of ω_c, however v_p is the phase velocity at ω_c, which cancels the explicit ω_c

Summary of time-frequency correspondance



- Frequency modes of the fs pulse are offset from $f_{n=0}=0$ by f_0

$$2\pi f_0 = \Delta\phi \; f_r$$

Parametric dependence

What laser parameters affect the comb frequencies f_r and f_0?

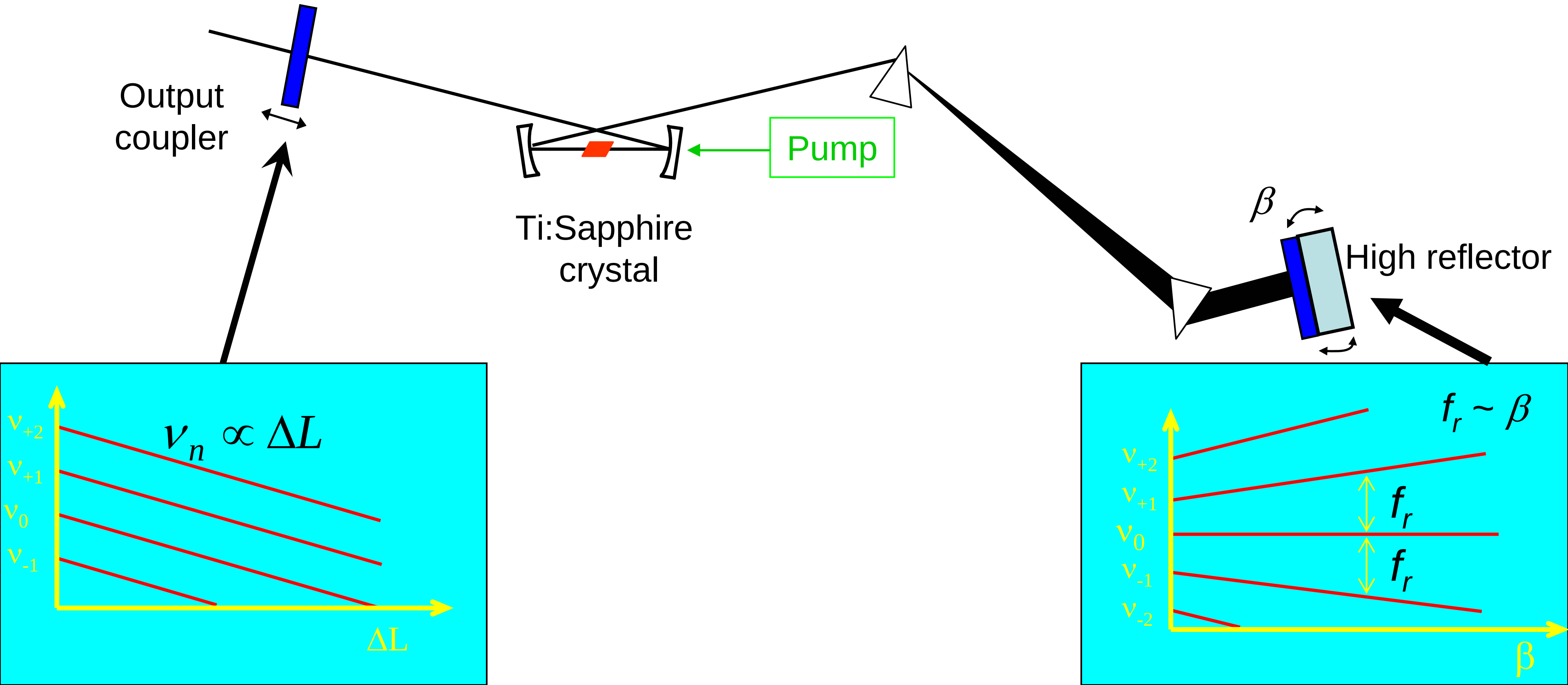
Parameter	f_r	f_0
Cavity length	Yes	2nd order (thru n_{eff})
Pulse energy	Thru Kerr effect	Thru finite response time of Kerr effect
Spectral shifts	Thru GVD	Thru $1/v_g - 1/v_p$
Alignment	w/ Prisms – yes Prismless – no	w/ Prisms – yes Prismless – no

How can we control the comb?

Through two of the parameters on the previous slide

- Need two – two degrees of freedom f_r and f_0
- Most commonly used:
  - Length – f_r
  - Pump power – combination of f_r and f_0

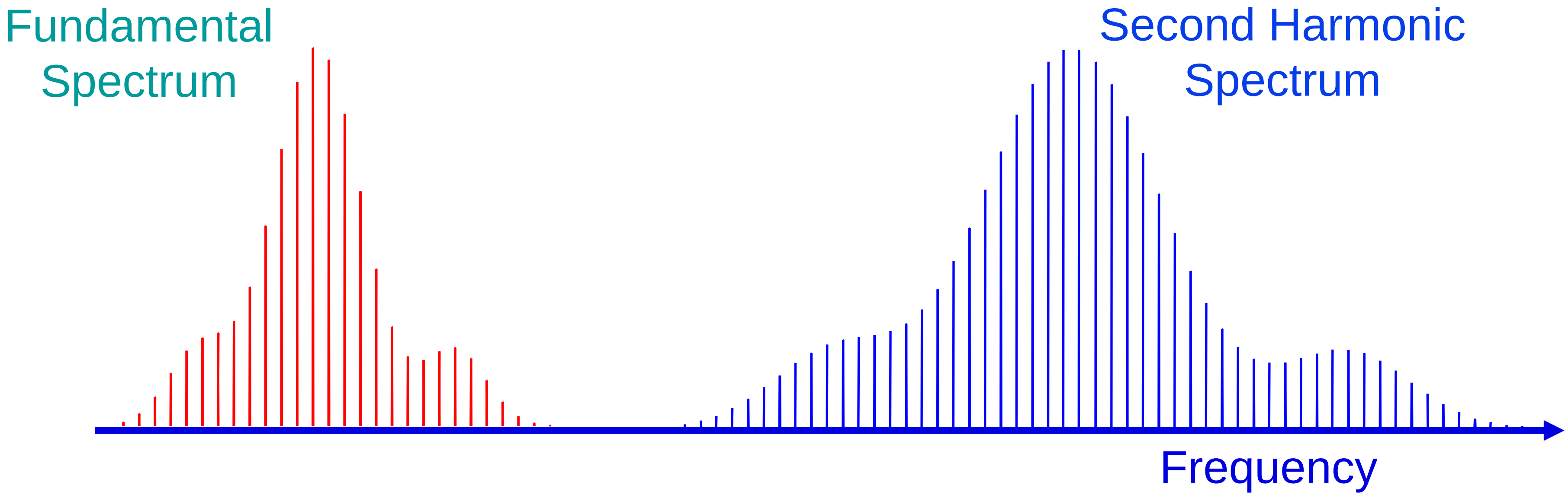
Also explicit control in prism laser through back mirror or prism/wedge



Second Harmonic Generation of modelocked lasers

Second harmonic of femtosecond pulses is really a misnomer – it is always sum frequency generation

This is particularly obvious for the comb spectrum



Effect of noise on the spectrum

The comb lines will have a finite width due to noise.

Technical noise sources include:

- Vibrations of mirrors
- Air currents
- Power fluctuations in pump laser

Fundamental/quantum noise cannot be eliminated:

- Driven by spontaneous emission from gain medium

The former can in principle eliminated, but not the latter.

Linewidth of a CW laser

The line-width, $\Delta\nu$, of a single-mode CW laser is given by the famous Schawlow-Townes formula:

$$\Delta\nu = \frac{4\pi h\nu}{P}\Delta\nu_c$$

ν – lasing frequency

P – power

$\Delta\nu_c$ – cavity linewidth

1/photon number

Physics: Spontaneously emitted photons from the gain medium have the wrong phase – results in phase diffusion of the coherent light

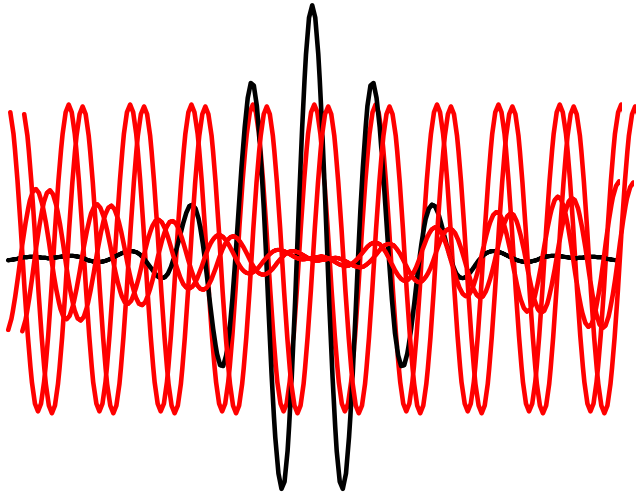
Does this formula apply to the individual comb lines?

No – they are **not** independent lasers

A.L. Schawlow and C.H. Townes, Phys. Rev. **112**, 1940-1949 (1958)

Effect of noise on a pulse

- Consider effect of noise on pulse, not independent cavity modes
- Decompose noise into:
  - Real vs. Imaginary
  - Even vs. Odd



Fundamental Pulse Parameters:

1) Amplitude

2) Phase

3) Time

4) Center Frequency

Equations of motion

Haus-Mecozzi perturbation theory results:

[H. A. Haus and A. Mecozzi, IEEE J. Quantum Electron. **29**, 983 (1993).]

$$T_R \begin{pmatrix} \dot{w} \\ \dot{p} \\ \dot{\phi} \\ \dot{t} \end{pmatrix} = \begin{pmatrix} -A_{ww} & 0 & 0 & 0 \\ 0 & -A_{pp} & 0 & 0 \\ A_{\phi w} & A_{\phi p} & 0 & 0 \\ -A_{tw} & -A_{tp} & 0 & 0 \end{pmatrix} \begin{pmatrix} w \\ p \\ \phi \\ t \end{pmatrix} + T_R \begin{pmatrix} S_w(T) \\ S_p(T) \\ S_\phi(T) \\ S_t(T) \end{pmatrix}$$

noise sources

Fluctuations from steady state in:

w - Intensity

ϕ - Phase

p - Center

t – local time

Physics:

A_{ww} – intensity returns to steady state due to gain saturation

A_{pp} – center frequency returns to peak of gain

$A_{\phi w}$ – Kerr effect: intensity dependent phase

$A_{\phi p}$ – phase velocity dispersion

A_{tw} – shock

A_{tp} – group velocity dispersion

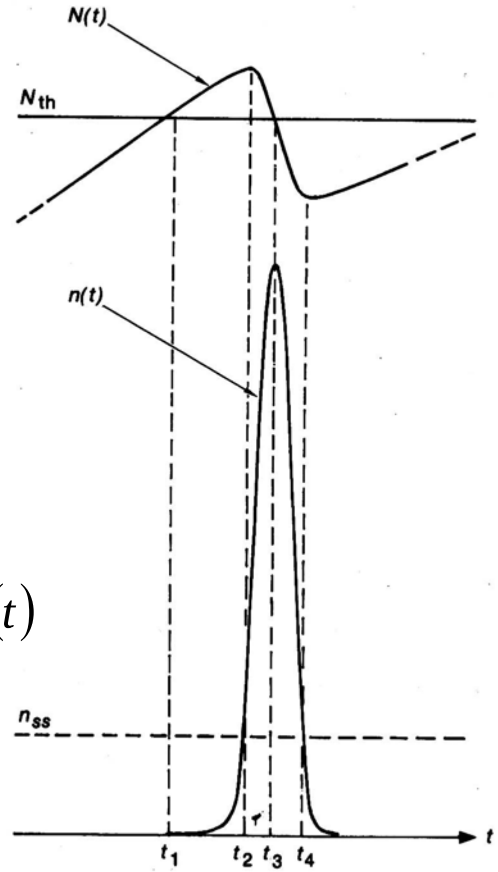
Turns out that neglecting gain dynamics is an over simplification:

Relaxation oscillations occur and are significant

Relaxation Oscillations

Relaxation oscillations, or spiking on turn on, occur because of the finite time it takes for the gain to saturate.

Think of as trading energy back-and-forth between light an inversion in gain medium



$$\frac{dn_p(t)}{dt} = KN_a(t)n_p(t) - \gamma_c n(t)$$

$$\frac{dN_a(t)}{dt} = R_p - \gamma_2 N_a(t) - KN_a(t)n_p(t)$$

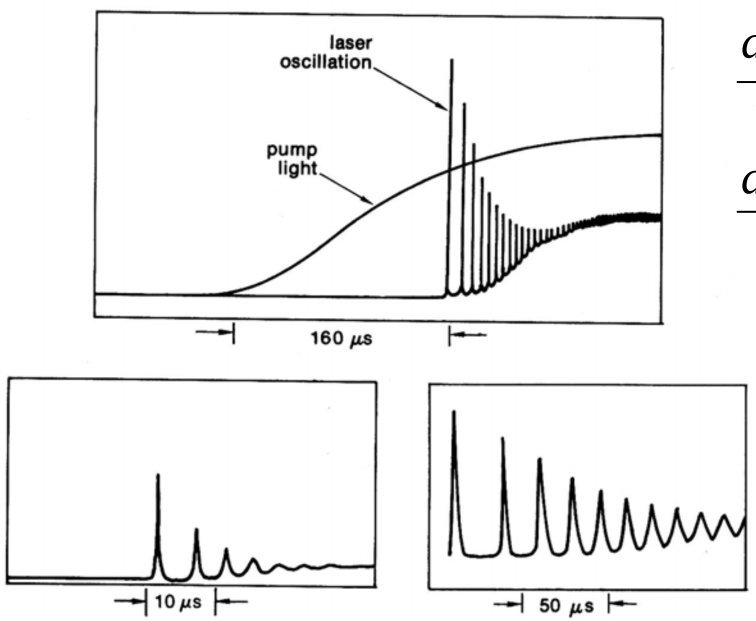
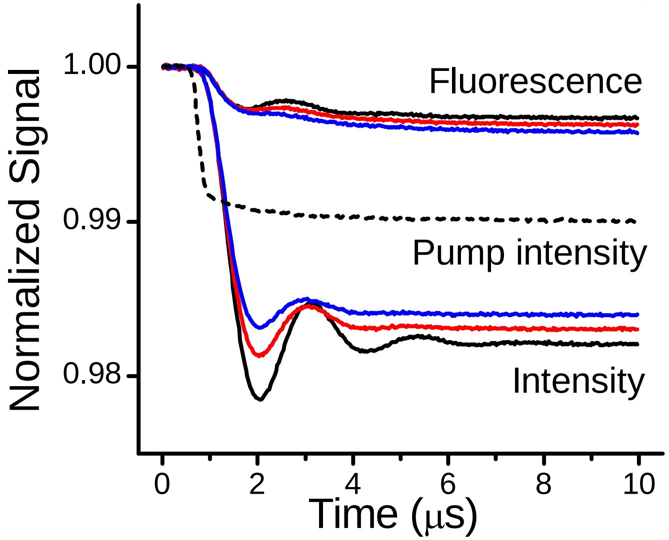


FIGURE 25.4
Spiking turn-on behavior in three typical Nd:YAG lasers.



Siegman, "Lasers" Chapter 25

Expanded Equations of motion

$$T_R \begin{pmatrix} \dot{w} \\ \dot{p} \\ \dot{g} \\ \dot{\phi} \\ \dot{t} \end{pmatrix} = \begin{pmatrix} -A_{ww} & A_{wp} & A_{wg} & 0 & 0 \\ A_{pw} & -A_{pp} & A_{pg} & 0 & 0 \\ A_{gw} & A_{gp} & -A_{gg} & 0 & 0 \\ A_{\phi w} & A_{\phi p} & A_{\phi g} & 0 & 0 \\ A_{tw} & A_{tp} & A_{tg} & 0 & 0 \end{pmatrix} \begin{pmatrix} w \\ p \\ g \\ \phi \\ t \end{pmatrix} + T_R \begin{pmatrix} S_w(T) \\ S_p(T) \\ S_g(T) \\ S_\phi(T) \\ S_t(T) \end{pmatrix}$$

noise sources

w - Intensity

ϕ - Phase

γ - Gain

p - Center

t – local time

These equations can accurately model the pulse dynamics in a mode-locked Ti:sapphire laser.

The matrix elements can be estimated from physical parameters (small signal gain, saturated gain, cavity lifetime, etc.) to within a factor of 2 or 3

or measured directly from the dynamics.